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TEST OF BOUNDED LOG-NORMAL PROCESS FOR OPTIONS PRICING

Semih Yön[1], Cafer Erhan Bozdağ[2]
[1,2]Department of Industrial Engineering, Istanbul Technical University, Macka Besiktas, 34367 Turkey

Abstract. *Bounded stochastic processes may be occupied for modelling the underlying price process in options pricing. Bounds can be included in some shallow markets by ceiling-floor price rule. It is useful to apply such a rule when an increase in the liquidity is needed. Log-normal process brings some bias on the premium of options. It is possible to reduce the bias by adding more parameters like jump diffusion, stochastic volatility or regime switching. As a result closed form solutions and numerical approximations suffer from increased dimension. Monte Carlo integration then appears to be unique solution for high dimensional calculations. However variance of the output of interest should be decreased in order to have confident results. The method of Importance Sampling can be used in an attempt to reduce error term. We test the bounded log-normal process with Importance Sampling Monte Carlo Simulation. Our analysis is based on the theory of variance reduction. Numerical results indicate that the risk neutral density should be substituted in the range of moneyness.*

Keywords: *Options pricing, bounded lognormal process, importance sampling, moneyness.*

Introduction

In developed financial markets firms and individuals seek new methods to minimize the risk arises from their transactions. Options allow investors to control the risk level when included to the portfolios. Huge amount of transactions in the options market makes options pricing one of the most attractive topics. Very famous Black and Scholes (1973) model lightened options pricing process. Their assumptions have been analysed enormously. In this study we test bounded log-normal process in terms of variance reduction capability.

It is impossible to specify the probability distribution of risky asset returns. Options pricing is based on a different probability space that calculations are done with respect to arbitrage free principle and risk neutral pricing. In this probability space asset price is simply an expectation of the discounted measure of its terminal value. The expectation is taken under an equivalent (semi) martingale measure which is a mapping of original probability distribution of the asset returns. Arbitrage free principle provides existence of an equivalent martingale measure which is not unique if the market is incomplete, Liu and Zhao (2013). It's assumed that logarithmic returns have an equivalent normal distribution in Black/Scholes model. Hence, distribution of the asset price becomes log-normal. Merton (1976) added price jumps to the log-normal process. However final model cannot prevent some bias on premium which increases with the maturity of the option. There are two main reasons for the bias. First one is market crashes not reflected by log-normal process with constant volatility. Hull and White (1987) introduced basic solution to stochastic volatility models excluding correlation between the volatility and spot price. Heston (1993) and Aït-Sahalia and Kimmel (2006) tried to find closed form solutions for general stochastic volatility models. Second bias comes from the market frictions such as transition costs and bid-ask spread. Longstaff (1995) calculated implied volatility of S&P index call options for two years and found the result that implied volatility of S&P index options has a smile pattern. This is known as volatility smile anomaly and studied by various authors like Rubinstein (1994), Neumann (1998) and Jackwert and Rubinstein (1996). Black/Scholes model imposes to observe the current underlying price from the market but Longstaff (1995) relaxed the underlying S&P index values and showed that it is more expensive to purchase the underlying asset from options market than the stock market. This is because of more transaction costs of options market. But Jackwert and Rubinstein (1996) showed that even the transaction costs remain constant the volatility smile happens to have different patterns for options with different underlying assets which proves that the only reason for the bias is not market frictions. Longstaff (1995) defined this basic assumption as martingale restriction. Estimating the implied index value and the implied volatility is the same as estimating the first and the second moments of the risk neutral underlying density which is assumed to be log-normal in most cases. Hence, diversification is mostly caused by the log-normal process itself. Neumann (1998) used two log-normal distributions as a mixed distribution to fit empirical data better. Neumann (1998) calculated parameters of the mixed log-normal distribution with least squares error technique. The analysis gets dependent to market events if any term is selected for the input data. The contribution of our study is that we do not use empirical data to test the risk neutral density. We carry out Importance Sampling Monte Carlo integration technique to the log-normal stochastic price process to evaluate the fair price of options.

Other advances to come up with the bias of Black-Scholes model are based on regime switching models. Bastani et al, (2013) study on American options with a radial basis collacation method. Boyle and Draviam (2007) studied on exotic options under regime switching model. Liu and Zhao (2013) deal with lattice methods for two underlying assets in regime switching model. Single risk-neutral density is not enough to represent the dynamics of option prices. Therefore randomly changing combination of Lévy processes included to the models. Brownian motion is the only Lévy process having continuous patterns. On the other hand, regime switching models with general Lévy processes are discrete realizations of the actual process whose states are determined by a continuous time Markov chain. Thus, it is allowed to specify long run equilibrium probabilities. The rate of return of the regime switching models with a number of parameters converges to the expected risk neutral rate.

The paper is organized as follows. Section 2 is devoted to the options pricing basis with log-normal underlying price process. Section 3 is concerned with Monte Carlo integration framework and Importance Sampling technique. Section 4 is devoted to the numerical study while numerical results are discussed in the sense of variance reduction capability. And concluding remarks are set at the end.

Options Pricing in Closed Form

Black and Scholes (1973) model is based on the main assumption of normal distributed logarithmic returns. The underlying asset price follows a geometric Brownian motion which is also called log-normal process. Then underlying price dynamics were reflected by a stochastic differential equation (SDE) as

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where $S_t \in Q^+$ is the spot price, μ is annual drift, σ is annual volatility of underlying and dW_t is the Wiener process. One easy way to show the derivation of this SDE is as follows: For one period, risky asset price can be expressed as

$$S_t = S_{t-1} u_t \tag{2}$$

where $u \in \mathbb{R}^+$ is a random variable which includes all economic information to change the price. Next step is to have logarithm of both sides and to start from the initial state

$$\log S_t = \log S_0 + \sum_{k=1}^{t} \log u_k \,. \tag{3}$$

In equation (3) all $\log u = \xi$ are selected normal random variables with $\xi \sim \mathrm{N}(\mu, \sigma^2)$ as imposed in Black and Scholes (1973). As a result logarithmic price ($\log S_t$) becomes a normal random variable since summation of normal random variables is still a normal random variable. A constant $\log S_0$ can be added to the summation while keeping normal property. The expectation and variance of the logarithmic price can be calculated from equation (3) where $\mathrm{E}[\log S_t] = \log S_0 + \mu t$ and $\mathrm{Var}[\log S_t] = \sigma^2 t$. Taking into consideration the stochastic normal random variable $X_t = \log \frac{S_t}{S_0}$, standard normal z can be expressed as in the following

$$z = \frac{X_t - \mu t}{\sigma \sqrt{t}}. \tag{4}$$

and equation (4) can be arranged as

$$\log S_t - \log S_0 = \mu t + \sigma \sqrt{t} z_i \tag{5}$$

where $z_i \sim \mathrm{N}(0,1)$ represents a random number for simulation trials. Then we have differential for both sides in equation (5) and get

$$d \log S_t = \mu dt + \sigma dW_t \tag{6}$$

where $\sqrt{t} z_i$ is substituted with dW_t since it is a random variable satisfying Brownian motion $\left(B(t) - B(0) \sim \mathrm{N}(0,t)\right)$. Further discussion about stochastic calculus can be found in Seydel (2006) and Benth (2004). Finally in equation (6) differential of logarithmic price is substituted with $d \log S_t = \frac{dS_t}{S_t}$ and after an easy arrangement equation (1) is found.

Parabolic SDE has a boundary at expiration time $t = T$ which serves as the option price. Payoff function for call option is $max(S_t - K, 0)$ where K is exercise price. When $S_t < K$ call option pays off zero. And payoff function for put option is $max(K - S_t, 0)$. The prices of call option and put option were calculated by solving SDE in Black and Scholes (1973).

Underlying price process is of high importance because it figures out the expected terminal value of the underlying and it specifies the payoff. Risk neutral pricing principles force the rate of return to be bounded in order to prevent any arbitrage opportunities. Nevertheless under low volatility conditions stochastic price process can be substituted with some alternatives. For example bounded log-normal process would be almost the same as the original in a stable market. If volatility is high it may work with sufficiently large bounds. Consequently the dimension of state space can be decreased.

Options Pricing with Monte Carlo Simulation

Option price (so called premium) is discounted value of the payoff under a risk neutral interest rate. Payoff is simply an expectation of the return determined by underlying price vector in arbitrage free environment. It is possible to generate future price vector with a random walk under an equivalent martingale measure. The price is calculated with the following formula for log-normal process.

$$S_{i+1} = S_i e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}z_i}, z_i \sim \mathrm{N}(0,1) \tag{7}$$

where r is the riskless interest rate and S_0 is observed from the market. Time frame Δt is set in years and might have $\Delta t = 1/252$ for working days per year in case of daily closing prices are simulated. When terminal value is identified with using equation (7) e.g European put option premium can be calculated with

$$V_p(S,T) = E^{LN}\left[e^{-r(T-t)}(K - S_T)^+\right]. \tag{8}$$

Monte Carlo Integration

Monte Carlo integration technique is widely used in derivatives pricing . Many problems in this area can be formulated as integrals over a single model distribution or highly multi-modal distributions in result of expectations like

$$\theta_f = E_f[q(\boldsymbol{x})] = \iiint_{R^d} q(\boldsymbol{x})f(x)d\boldsymbol{x} \tag{9}$$

where $q(\boldsymbol{x})$ is a real valued function of interest. The notation θ_f, E_f denotes that the expectation is taken with respect to density $f(\cdot)$ which belongs to the d-dimensional probabilistic state space Ω. If it is hard to find a closed form solution to equation (9) Monte Carlo simulations can be warranted to provide approximate results. Simulations driven by random inputs will produce random outputs. And those random outputs are the estimation of the exact results. The accuracy of this estimation strongly depends on quality of sampling which can be improved in two ways:

- increasing the cardinality of sampling or,
- introducing some kind of selection rules that make it more representative.

The first choice is limited way whereas the second requires to apply some special techniques like Importance Sampling (IS) which is explained in the next section.

Importance Sampling as a Variance Reduction Technique

In Monte Carlo applications variance of the output random variable should be reduced without disturbing its expectation which means smaller confidence intervals. Importance Sampling (IS) introduces definite selection rules to generate most likely configurations to obtain more accurate values of statistical averages. Certain values of the input random variables in a simulation have more impact on the parameter being estimated than others. If these important values are emphasized by sampling more frequently, then the estimator variance could be reduced. Yön and Goldsman (2006) deal with some useful biasing methods. Hence, the basic methodology is to choose a new distribution which encourages the important values. This use of a biased distribution will result in a biased estimator. However, the simulation outputs are weighted to correct for the use of the biased distribution, and this ensures that the new IS estimator is unbiased, Broadie and Glasserman (1997). IS can be carried out as in the following way:

$$\theta_g = E_g\left[q(\boldsymbol{x})\frac{f(\boldsymbol{x})}{g(\boldsymbol{x})}\right] = \iiint_{R^d} q(\boldsymbol{x})\frac{f(\boldsymbol{x})}{g(\boldsymbol{x})}g(x)d\boldsymbol{x}\,. \tag{10}$$

Random samples are generated from $g(\cdot) \in \Omega$ which is called IS density. $g(x)$ enables to calculate the correction factor $\frac{f(x)}{g(x)}$ which is sometimes called weight function. Based on sample weights accumulated during sampling the correction factor compensates for statistical fluctuations and lead to a lower variance. In equation (10) the IS density $g(x)$ should assign higher probabilities to important region while holding $\theta_f = \theta_g$, Yön (2007). Then the estimator can be calculated as

$$\widehat{\theta_g} = \frac{1}{N}\sum_{i=1}^{N}\left(q(\boldsymbol{x_i})\prod_{j=1}^{d}\frac{f(x_j)}{g(x_j)}\right) \tag{11}$$

where N is the replication number and d is the dimension of the multivariate underlying distribution. Note that $f(\cdot)$ and $g(\cdot)$ are two independent densities. Finally Mean Squared Error (MSE) of the estimator is calculated with formula

$$MSE = \frac{\sum_{i=1}^{N}\left(\theta_i - \widehat{\theta_g}\right)^2}{(N-1)}\,. \tag{12}$$

The successful IS density leads lower possible MSE. This implies that underlying stock dynamics can be represented better with an alternative IS distribution. Detailed features of IS densities is given at Yön (2007) and Broadie and Glasserman (1997).

Numerical Results

We test bounded log-normal process from a variance reduction point of view by nominating Importance Sampling (IS) technique. We first carry out crude Monte Carlo simulation and then run IS for the same input variables. Bounded process is applied as in a way that the trading price is allowed to change in 10% limits for up and down directions with respect to previous day's closing price. We try to implement a number of underlying distributions as IS densities like Gama, truncated Pareto and mixture of log-normal distributions. Numerical results indicate that it is possible to have high variance reduction for a wide range of moneyness. We fix the input parameters as $T = 1$ year, $r = 10\%$, $K = 50$ and relaxed the spot price in the range of $S_0 \in [30,70]$ with unit increments. We have two groups of runs according to volatility in the market. We have small volatility $\sigma = 20\%$ and large volatility $\sigma = 80\%$ in order to observe the effect of high and low variation. We develop an efficient C program that the simulation with one million replications takes a few seconds. Figure 1a and Figure 1b show the graphs of call option and put option price changes for high volatility. We have results in line with expectations. Bounds make call option price lower because the increase in the underlying price is limited with bounds. Expected call payoff is reduced. Zero is a natural lower bound for all assets but payoff for call is reduced with upper bound. In contrast, put option price increases with bounds because the probability of put option being worthless is reduced with upper bound. And payoff for put increases. General trend with low volatility is similar to high volatility case but the results are so close. The difference in values of three lines for low volatility case is so little that cannot be displayed in a large scaled graph.

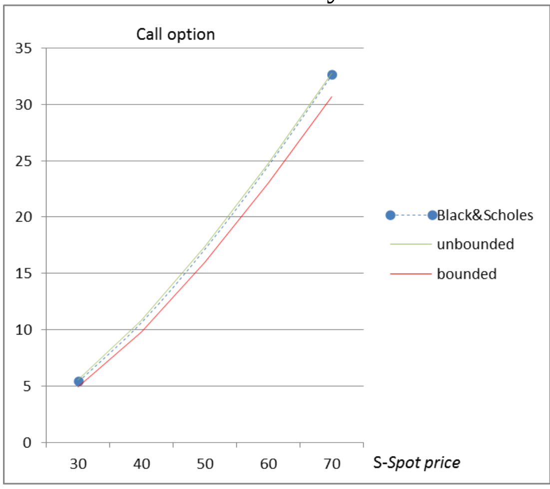

Figure 1a: Call option price is graphed with respect to spot price. $K = 50$ is fixed. Unbounded process and Black/Scholes prices are very close whereas bounded process gives lower prices for call options.

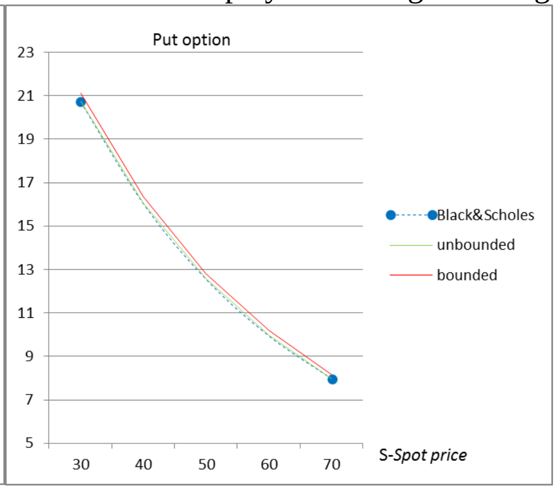

Figure 1b: Put option price is graphed with respect to spot price. $K = 50$ is fixed. Unbounded process and Black/Scholes prices are again very close. However bounded process gives higher prices for put options.

Numerical results also show that it is possible to have higher variance reduction for out-the money options. Intuitively this suggests to have different underlying distributions for different moneyness regions. Then it would be possible to reflect the underlying price dynamics better.

Conclusions

Economic data influence prices a lot, contribution of our study is that we do not use term dependent empirical data. We used variance reduction technique and simulation to test the lognormal process. The possibility of high variance reduction shows that original risk neutral measure of log-normal distribution cannot completely reflect the underlying price dynamics. We used importance sampling in our analysis. The basic idea is to compute a correction factor to the importance sampling estimates. Better alternatives could be found by easy combination of continuous distributions. Bounded lognormal process is suitable for some markets. Both approaches could be better in the form of a risk neutral density for different moneyness regions.

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