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On Relation between No-Arbitrage Pricing Principle and Modigliani-Miller Propositions

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ABSTRACT

An extension of Merton's (1974) model (EMM) taking account of the firm's payments and generating a new statistical distribution for the firm value is suggested. In an open log-value space, this distribution evolves from the initially normal to negatively skewed one. When payments are zero or proportional to the firm value, EMM turns into the Geometric Brownian model (GBM). We show that Modigliani-Miller Propositions (MMPs) and the no-arbitraging principle (NAP) result from the use of GBM with no payments. For a firm with payments, MMPs hold for short times and are false for time intervals exceeding a year. In contradiction with MMPs, the asset structure affects the firm value at the perfect market, and at the market with taxes, debt decreases the firm value even when there are no bankruptcy costs. NAP always holds for the entire market for short time deals. For long-term investments, the firm's mean year returns decline in time intervals whose length depends on the firm's initial conditions and its business environment. In these conditions, NAP does not hold for the whole market, but it temporarily holds for individual stocks as far as the mean year returns of the firms issuing them remain constant and fails when the mean year returns begin to decline.

Introduction and literature review

The question of what is a mathematical description of the firm value is among the most significant matters in the theory and practice of financial economics. So far, two statistical distributions are most often used to describe the firm value: the normal distribution and lognormal one. Bachelier (1900) suggests that the stock price follows the Brownian motion model, which means the firm value distribution normality. Kendal (1953) and Onsager (1959), through statistical analysis of large volumes of economic data, show with sufficient accuracy and precision that the relative firm returns behave like normally distributed independent random values. Samuelson (1965) generalizes the findings of Kendal and Onsager in the "economic exponent" implying the lognormal distribution for the firm value. We show that Modigliani and Miller in their celebrated "Theorem of Irrelevance" (MM Proposition I, 1958), proving that there is no relation between the firm's capital structure and its mean returns and value, implicitly use the lognormal distribution. Markowitz (1952), in his seminal paper on portfolio investment theory, also assumes implicitly that firm returns are normally distributed, which implies the value log-normality. Black and Scholes (1973), in their classic option pricing study, use a geometric Brownian model (GBM) that leads to the lognormal distribution of option prices. Merton (1974), considering the probability of corporate default, suggests a more general model for the firm value development, introducing debt and dividend payments into the model. However, he restricts his analysis with the GBM case only, assuming that "while options are highly specialized and relatively unimportant financial instruments [. . .] the same basic approach could be applied in developing a pricing theory for corporate liabilities in general" (1974, p. 449). Leaving aside that this statement contradicts his general model, it looks most surprisingly considering dramatic differences between the options on the one hand, and the corporate liabilities in general, such as stocks or bonds, on the other. The option is a short-living financial instrument whose existence is guaranteed within its expiration period, making typically 60 or 90 days. The short expiration period makes the option insensitive to changes in the state of the firm, which has issued the corresponding corporate liability. The stocks and

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bonds are potentially long-living instruments, which can default, theoretically, at any moment. Their market position depends strongly upon the state of the firm issuing them. Merton applies GBM in his analysis of the price of a zero-coupon bond of face value D and maturity T issued by the firm, which defaults if its assets occur to be less than its outstanding debt at the time of debt maturity. Considering these bonds independently of the firm's state, he naturally comes to a result identical to that of Black and Scholes. Merton interprets this identity as a proof of the general validity of GBM and his option hypothesis.

This opinion is shared worldwide in the financial academic community; below we give a brief list of articles based on GBM. In a comprehensive review on dynamic structural models, Strebylaev and Whited (2012, pp. 4-5) state that "they (dynamic structural models) start with the acknowledgement that any claims on corporate cash flow streams are derivatives on underlying firm value or firm cash flows. This means that we can apply option pricing methods to value these claims." The authors insist that GBM can be used for describing the "stock price in the Black-Scholes model; firm value, firm cash flows, or prices of firm output and input in other corporate finance settings." Sundaresan (2013, p. 21) asserts that "since its publication, the seminal structural model of default by Merton (1974) has become the workhorse for gaining insights about how firms choose their capital structure, a "bread-and-butter" topic for financial economists."

To make the description of the firm more realistic, Black and Cox (1976) improve the Merton model introducing a threshold triggering default when the firm's assets hit the threshold. Since that time, the structural model becomes the most popular instrument of study in different fields of economics and finance. Here we briefly list some of the directions of investigation using the structural models.

Leland (1994) makes the next major step introducing explicitly into the model corporate taxes and bankruptcy costs to determine the optimal capital structure for a firm with a debt of infinite maturity. Leland and Toft (1996) revisit the problem of the optimal capital structure, extending the Leland's results to the case of debt with arbitrary maturity. Collin-Dufresne and Goldstein (2001), Goldstein et al. (2001), Ju et al. (2005), and Titman and Tsyplakov (2007) explore the issue of dynamic capital structure. Ju and Ou-Yang (2006) offer a structural model for the dynamic capital structure with stochastic mean-reverting interest rates. Hackbarth et al. (2007) study the difference between private (bank) and public debt in debt renegotiation. Francois and Morellec (2004) and Broady et al. (2007) consider a departure from the absolute priority rule and debt renegotiation under Chapter 11. Abel and Eberly (1996) and Bloom et al. (2006) apply a structural model for solving the problem of the optimal investment of a firm that faces costly reversibility under uncertainty. Chen (2010) and Bhamra et al. (2010) analyze relations between the macroeconomic state of the economy and the intensity and scale of defaults. Barclay and Smith (1995), Barclay et al. (2006) investigate relations between the firm's investment decisions and its financing decisions, etc. The literature on structural model (GBM) applications includes many dozens of names and titles, and, unfortunately, there is no way even to mention all of them here. One can find a more detailed discussion of structural model applications in excellent reviews of Laajimi (2012) and Sundaresan (2013). Since the 1990s, the *risk-neutral approach* becomes a dominating method of analysis of the firm's credit risks and default probabilities, significantly simplifying this analysis (e. g. Strebulaev & Whited, 2012).

However, the default probabilities predicted with a GBM-generated normal log-value distribution occur much lesser than the default frequencies observed in practice; it means that the real log-value distribution has heavy tails, or is negatively skewed. Since the end of the 1980s, theoretical studies on the firm value distribution and default probabilities split into two main directions. The *jump-diffusion processes* (JDPs) supplementing GBM with Poisson jumps in the firm value make the first one (e. g. Zhou 1997; Hilberink & Rogers 2002; Kou 2002; Chen & Kou 2009). As assumed, the jumps represent a market reaction to new information about the firm, and dominating leaps down provide for the desired negative skewness to the firm value distribution. Estimation of fixed jump parameters, their intensity and mean length, make a separate problem usually resolved with calibrated models (Leland 2006).

Giesecke and Goldberg (2008) use a structural model of credit risks to show that informational asymmetries can induce an event premium for the abrupt changes in security prices that occur at default. If the public investors are unable to observe the threshold asset value at which the firm's management liquidates the firm, then they face a sudden default risk as they cannot discern the firm's distance to default. Technically, the authors suggest another kind of JDP model adding an extra jump risk to a low diffusion risk. To apply the martingale technique to the problem solving, the authors use GBM with no payments and jumps in the firm value down to a default line through a random location of that line. However, this abrupt change in the security price, allegedly caused by information asymmetry between the firm's management and the public investors, admits another explanation. A real default probability generated by the negatively skewed firm value distribution is higher than its GBM-estimation used by the firm's management as well as public investors. This difference creates an unexpected risk coming, as they believe, due to information asymmetry. Actually, the JDP solution tries to compensate for the difference between the higher default probability for firms with payments and its GBM-estimation. As we show further, the real distribution for the firm with payments has negative skewness growing over time that cannot be provided by JDPs with fixed jump parameters.

The second line of investigations trying to step outside the distribution normality consists of *stochastic volatility processes* affording heavy distribution tails (e. g. Hull & White 1987; Melino & Turnbull 1990; Nicolato & Venardos 2003). This group of models is used mainly for option pricing; we do not consider them here.

Another way to take heuristic account of the value distribution skewness is implemented in so-called *calibrated models*. We show shortcomings of this class of models on an example of Moody's KMV (Bohn, 2006). To introduce negative skewness to the log-value distribution, the model uses an extensive database of real defaults for estimating default probabilities and the loss distribution at a time horizon of one year. The model applies to publicly traded firms for whom market values are known. To determine a firm's current state, the model uses GBM with no payments to calculate the distance-to-default (DD) as a height of the log-value mean over a default line measured in standard deviations. Then using the database, the model determines a share of firms with that DD who have defaulted within a year. This share has got the name of Expected Default Frequency (EDF) and is a rough estimate of the intensity of default probability (IPD, see Section 1) at a time horizon of one year. Despite its popularity, the model suffers from serious drawbacks typical for calibrated models. First, the assumption that EDF is a function of DD only is wrong. Two firms having the same DD at some time can have different IPD values because the real distribution depends on a state of the firm and its business environment (the debt leverage, interest rate, inflation rate, taxation rate, etc.). Second, for credit risk estimation objectives, a creditor wants to know the probability of borrower's default at a horizon of the credit maturity. The credit maturity can achieve decades while Moody's KMV works at a time-horizon of one year only. The natural conclusion from all said above is we need a more accurate and precise model for the firm value.

In this paper we take an attempt to present such a model (we call it the Extended Merton model, or EMM for short) taking account of the firm's payments and breaking off any connection with GBM following from the false Merton's analogy between the firm and the option. We show that the EMM-firm and the GBM-firm have very different characteristics both at the firm level and at the market level. The GBM-firm remains "ever young", keeping the time-invariant mean year returns and volatility and producing overoptimistically low default probabilities. On the contrary, the EMM-firm by and by "grows old": its mean year returns decrease after some time depending on the firm parameters, its volatility and negative skewness continuously grow contributing to the default probability. On the market level, GBM-firms admit the time-independent risk-neutral measure, risk-neutral probabilities, and no-arbitrage pricing principle effective for the entire market. The market consisting of EMM-firms does not have the risk-neutral measure independent of time and for the whole market. However, each firm at the market can have its risk-neutral measure for some time determined by parameters of the firm and its business environment. The implication from this is that the no-arbitraging property becomes a feature of the firm, holding only for some limited time. Risk-neutral probabilities exist as far as the firm's mean year returns remain constant; thus, the risk-neutral approach is legitimate for safe firms only. One cannot use this approach for estimating credit risks and finding the firm's default probabilities.

We also show that Modigliani-Miller Propositions follow from the perfect market assumptions joined with an implicit assumption that the firm value meets the GBM-distribution with no payments at all. We show that none of MM Propositions holds when the firm's payments are proportional to the firm value, although the firm value distribution remains lognormal. The Propositions do not hold good also for EMM-firms whose payments make an arbitrary function of time, and the firm value distribution becomes negatively skewed. Because debt always decreases the mean after-tax value of the firm, the mean after-tax value of the unleveraged firm is higher, not lesser than the mean after-tax value of the identical leveraged firm. The last implies that MM Proposition III and the trade-off theory following from it are wrong; hence, all conclusions based on that theory are incorrect. For business practice, it means that long-term investors (such as pension funds, mutual funds, banks, etc.) led by recommendations based on the dominating theories, suffer supplementary losses. Under such theories, we understand GBM, the risk-neutral technique, universal no-arbitrage pricing principle, Modigliani-Miller Propositions, and their consequences, like the trade-off theory.

The rest of the paper has the following structure. Section 1 presents a continuous-time stochastic model estimating the probability that a firm with payments will meet in its development with financial difficulties that bring about the firm's default. We show that the firm value distribution is negatively skewed, and reveal a dependence of the default probability on factors of the firm and its business environment. We demonstrate that the GBM-solution and the EMM-solution are very different, and trace this difference to the marginal deterministic case.

In Section 2, we analyze qualitatively the properties of the EMM-firm's value distribution. We show that the no-arbitrage pricing principle is consistent only with the markets described by GBM, no payments. This model does not consider the firm's payments, such as fixed costs, debt payments, taxes, or dividends. The firm is equivalent to a self-financing portfolio (Harrison & Krebs, 1979). For the market of firms with payments, the no-arbitraging principle becomes a time-dependent characteristic of individual stocks and the firms issuing them, rather than a feature of the market as a whole. We show that the validity of the no-arbitrage pricing principle does not depend on the way of payments, continuous or discrete. At the end of Section 2, we present the results of computer modeling of the firm value development for various initial conditions supporting our qualitative analysis.

In Section 3, we analyze the Modigliani-Miller Propositions (MMPs) and their relation to the no-arbitrage pricing principle. Varian (1987) argues that "the importance of arbitrage conditions in financial economics has been recognized since Modigliani and Miller classical work on the financial structure of the firm. […] Modigliani and Miller's proof of their Propositions used an ingenious arbitrage argument. Many important results of financial economics are based squarely on the hypothesis of no-arbitrage, and it serves as one of the most basic unifying principles of the study of financial markets". The fundamental importance ascribed to MM Propositions (Modigliani & Miller, 1958; 1961; 1963) and the no-arbitrage pricing principle (Cox, Ross, & Rubinstein, 1979; Harrison & Krebs, 1979; Harrison & Pliska, 1981) for the theory and practice of financial economics make us scrutinize both results most carefully.

MM Propositions I and II are formulated for the firms acting at the perfect market. We show that MM Propositions use an additional implicit assumption that the firm value has a lognormal distribution corresponding to the case GBM, no payments. For the model with payments, the first MM Proposition (the asset structure does not affect the firm value) is correct only conditionally. It is valid for short-term deals only, when the firm's liabilities such as stocks or bonds are bought and then soon resold in an attempt to profit on the price difference. For times longer than a year, the mean year returns of the unlevered firm surpass the mean year returns of the identical levered firm, and the unlevered firm value surpasses the levered firm value. (After Modigliani and Miller, we consider the levered firm identical to the unlevered firm in any respect, but the capital structure.) Propositions II and III are always wrong because they make their conclusions for time intervals exceeding a year, but such time intervals are outside the domain of GBM validity.

Model description

In his seminal work, Merton (1974) introduces a continuous-time equation describing the development of the firm value in a random environment:

$$dX = (\alpha_0 X - P)dt + C^{1/2} X dW, \quad X(0) = X_0, \tag{1.1}$$

$$P = DIV + DP. \tag{1a.1}$$

Here $X(t)$ is a firm market value at time t, constant α_0 is an instantaneous expected return on the firm per unit time, P is the total dollar payouts by the firm per unit time to either its shareholders or liabilities-holders (dividend DIV or interest DP payments) per unit time, constant $C > 0$ is the instantaneous variance of returns, W is a Wiener process representing a cumulative effect of normal shocks. (For the sake of consistency with the further discussion, we use our own symbols for the variables and parameters in the model keeping the original Merton's interpretation of the symbols. One should note, however, that C is not the variance of returns, but the rate of variance growth and $C^{1/2}$ is the process volatility.)

Merton solves a special case of this equation with $P = 0$. When $P = 0$, or $P = \delta_0 X$, δ_0 is constant, Eq. (1.1) transforms into the geometric Brownian model, GBM:

$$dX = \lambda X dt + C^{1/2} X dW , \qquad \lambda = \begin{cases} \alpha_0, & P = 0 \\ \alpha_0 - \delta_0, & P = \delta_0 X \end{cases}. \tag{2.1}$$

From Eq. (2.1) it follows that GBM requires unrealistic conditions for its validity. The condition $P = 0$ (Black & Scholes 1973, Merton 1974, Black & Cox 1976, Leland 1994a, etc.) means that a firm makes no payments at all, which is impossible. The condition $P = \delta_0 X$ makes the payment P another variable cost that just affects the effective expected rate of returns (e. g. Leland 1994b, Leland & Toft 1996, Goldstein et al. 2001, Huang & Huang 2012). The relation $P = \delta_0 X$ could be considered as more or less acceptable when payment P consists of dividends only, but when payment P includes fixed costs and/or debt payments, the relation $P = \delta_0 X$ becomes unreal. Payment P has its schedule hardly related to changes in the firm size and value, and the payment certainly does not follow the firm value when it drops to a default line.

A GBM solution is a lognormal distribution

$$U(X,t) = (2\pi\sigma^2)^{-1/2} X^{-1} \exp[-(\ln X - H)^2 /(2\sigma^2)], \tag{3.1}$$

$$H(t) = H_0 + Rt\,,\ \ \sigma^2(t) = \sigma_0^2 + Ct\,,\ \ R \equiv \lambda - C/2\ .$$

In the original Merton model, a firm defaults only if its value is less than the firm's outstanding debt at the time of debt maturity. Black and Cox (1976) improve this shortcoming introducing a threshold (a default line) triggering default any time when the firm value hits the default line. An excellent introduction to modern methods of credit risk estimation one can find in (Crouhy et al., 2006; The Credit Market Handbook, 2006).

Because of the role, which the firm value has in financial economics, the general Merton model and its solution are crucial for the financial economics theory and practice. However, before studying the model, we revise it because Merton's interpretation of payments is too short. The revised model considers a firm that manufactures and markets its production or rendering services at the market subject to random shocks having a normal distribution. The market shocks affect the firm value with the intensity $C^{1/2}$. While doing business, a firm makes various payments. Some of them are closely related to the manufacturing and marketing of the firm's goods; they are proportional to the firm value (variable costs) and can be taken into account by adjusting the rate of returns α_0. Other payments secure the very firm's presence in business. Such expenses include fixed costs (*FC*), corporate taxes (TAX), dividends (*DIV*), and debt payments (*DP*), all per unit time. Thus, one can write for *business securing expenses* (*BSEs*)

$$P = FC + DP + TAX + DIV, \tag{4.1}$$

$$P(t) = P_0\,\pi(t), \qquad P(0) = P_0 > 0, \quad \pi(0) = 1.$$

Here $P(t)$ is an arbitrary function of time, P_0 is a positive constant. The time dependence of *FC* and *DP* reflects changes in business conditions; *TAX* and *DIV* depend on their rates and year returns. Hereafter the process (1.1), (4.1) is referred to as an Extended Merton model or EMM for short.

Equation (1.1) for random variable $x = \ln(RX/P_0)$ by Ito's Lemma transforms to

$$dx = R(1 - \pi(t)e^{-x})dt + C^{1/2}dW, \tag{5.1}$$

$$x(0) = x_0 = \ln(RX_0/P_0), \qquad R \equiv \alpha_0 - C/2\,. \tag{5a.1}$$

Writing a Fokker-Plank equation for Eq. (5.1), one comes to an equation for the probability distribution $V(x,\,t)$, or x-distribution; V_y is a partial derivative over a variable y:

$$V_t + R\left(1 - \pi(t)e^{-x}\right)V_x - 0.5\,CV_{xx} + R\pi(t)\,e^{-x}V = 0\,. \tag{6.1}$$

The initial condition is

$$V(x,\,0) = V_0(x; H_0, \sigma_0^2)\,, \tag{7.1}$$

$$H_0 = \langle x(0)\rangle = \int_{-\infty}^{\infty} xV(x,0)dx\,, \quad \sigma_0^2 = \left\langle (x - H_0)^2\right\rangle = \int_{-\infty}^{\infty} (x - H_0)^2 V(x,0)dx\,,$$

where $V_0(x; H_0, \sigma_0^2)$ is a normal distribution.

For analysis of the credit risk problem, one must add a boundary condition implying that a firm comes to default when its value falls to X_D ($0 < X_D < X_0$)

$$V(DL, t) = 0, \quad DL = \ln(RX_D/P_0) . \tag{8.1}$$

If X_D is an outstanding debt as it is in (Black & Cox 1976), Eq. (8.1) is an *exogenous* constraint. If the firm has no debt, there is another constraint. A BSE share in the expected year returns is

$$P_0/(R\langle X_0\rangle) = \exp(-H_0 - \sigma_0^2/2). \tag{9.1}$$

For $H_0 \geq 0$, this share is less than one, while for $H_0 < 0$, it exceeds one, and the firm pays out more than it earns. The line $x = 0$ separates a profitable business from its failure. In this case it is reasonable to introduce a soft *endogenous* boundary

$$V(DL, t) = 0, \qquad DL = 0, \tag{10.1}$$

and watch the probability of crossing this line. Below the line $x = 0$, the firm can continue its activities, only selling some other of its assets. The nature of this boundary is close to the nature of the boundary introduced by (Kim et al. 1993) when a firm defaults if it runs out of cash.

The boundary conditions (8.1) and (10.1) can be joined as

$$V(DL, t) = 0, \quad DL = max(0, \ \ln(RX_D/P_0)) . \tag{11.1}$$

A solution of the problem (6.1), (7.1), (11.1) is the log-value distribution for a firm in financial distress; it is denoted as $\hat{V}(x, t)$. If one knows a solution $V(x,t)$ in the open space, then a solution of the boundary problem (6.1), (7.1), (11.1) can be written as

$$\hat{V}(x,t) = V(x,t) - V(2DL - x, t) . \tag{12.1}$$

The probability distribution drops to zero at the default line $\hat{V}(DL,t) = 0$, and the intensity of default probability $IPD(t)$ is

$$IPD(t) = 2\int_{-\infty}^{DL} V(x,t)dx . \tag{13.1}$$

The first three statistical moments are calculated along with the probability distribution $V(x,t)$:

$$H(t) = \int_{-\infty}^{\infty} xV(x,t)dx , \quad VAR(t) = \int_{-\infty}^{\infty} (x-H)^2 V(x,t)dx , \ S(t) = \int_{-\infty}^{\infty} (x-H)^3 V(x,t)dx . \tag{14.1}$$

$S(t)$ proportional to distribution skewness shows the development of distribution asymmetry. The main objective of any credit risk analysis is estimating the firm's default probability over a chosen time interval (e. g. the debt maturity)

$$PRD(t_S, T) = \int_{t_S}^{t_S+T} IPD(t)dt , \tag{15.1}$$

here t_S is a moment when the credit is issued, $t_S + T$ is a moment of debt maturity, and $PRD(t_S, T)$ is the default probability over credit maturity period T. In this paper, $t_S = 0$.

Now let us compare the EMM-solution of Eq. (5.1) - (5a.1) and the GBM-solution of Eq. (2.1) in the *deterministic case* free of the random shocks, that is (α and P are constants)

$$dX = (\alpha X - P)dt\,, \quad X(0) = X_0\,, \tag{5.1}$$

$$dX = \alpha X dt\,, \qquad X(0) = X_0\,. \tag{2.1D}$$

A solution of Eq. (5.1D) is $X_E(t) = P/\alpha + (X_0 - P/\alpha)e^{\alpha t}$. Depending on X_0 -value, there are *three* branches: a branch exponentially rising from the point X_0 if $X_0 > P/\alpha$, a constant branch $X_E(t) = P/\alpha$ if $X_0 = P/\alpha$, and a branch exponentially declining from the initial point X_0 to zero: $X_E(t) = P/\alpha - |X_0 - P/\alpha|e^{\alpha t} \geq 0$ if $0 < X_0 < P/\alpha$. The line $X_E(t) = P/\alpha$ is the line of unstable equilibrium; any deviation from that line sets $X_E(t)$ onto a rising or declining path. On the other hand, eq. (2.1D) has *only a rising* solution $X_G(t) = X_0 e^{\alpha t}$.

Despite the fact that for solving a stochastic problem (5.1) one uses the logarithmic variable $x = \ln\left(RX / P_0\right)$, and for a deterministic problem (5.1D) it is better to use the firm value *X*, there are many parallels between the stochastic models (5.1) and (2.1) and the deterministic models (5.1D) and (2.1D). $X_E(t)$ approaches $X_G(t)$ from below when the ratio $P/(\alpha X_0)$ tends to zero. We see the same picture for the rising parts of the log-value mean *H*(*t*) for various initial H_0 values in the stochastic model (Fig. 1). The equilibrium line $X_0 = P/\alpha$ strictly corresponds to the initial position of the line of unstable equilibrium $x = EQ(0) = 0$ in the stochastic model (5.1). The static line $X_0 = P/\alpha$ and the dynamic line $x = EQ(t)$ separate the rising and falling particles (see the next Section). Over these lines, the paths *H*(*t*) and $X_E(t)$ rise with rates limited from above both in the firm value space and in the log-value space. Below these lines, the paths have negative rates tending to infinity in the log-value space when *x* or lnX_E tend to negative infinity. For the deterministic model, this follows from the fact that $X_E(t)$ achieves zero in a finite time. Zero in the firm value space corresponds to negative infinity in the log-value space; thus, to achieve it in a limited time, the rate in the log-value space must tend to infinity when *x* or ln(X_{EM}) approaches negative infinity.

The most dramatic difference between the stochastic and deterministic models is a diffusion force appearing in the stochastic model as a result of the Brownian motion. In the deterministic case, all particles move along their unique differentiable paths. These deterministic paths do not intercept or touch each other. Random shocks make once deterministic paths become the non-differentiable ones and shuffle them; the Brownian motion arises. It divides all mean trajectories *H*(*t*) of the particles into two groups (the first rising then falling trajectories, and the trajectories declining from the start). In the deterministic case, there are three groups of paths, including rising, declining, or horizontal paths. Another difference is a lesser expected drift rate in the stochastic case ($R = \alpha - C/2$ against α).

The smaller the shocks in the stochastic model, the longer the behavior *H*(*t*) remains similar to behavior of ln(X_{EM}) until the moment when the accumulated skewness makes *H*(*t*) decline. A solution of the (2.1D) problem in the log-value space remains a straight line all the time $lnX_G \equiv H_G(t) = H_0 + \alpha t$, which is identical to the behavior of the GBM-solution (3.1). As we see, EMM-solution and GBM-solution differ significantly even in the deterministic case.

What type is *x*-distribution, and how does it depend on the mode of payment?

For random variable $x = \ln\,(RX/P_0)$, R is fixed, $\pi(t) \equiv 1$, no debt, Eq. (5.1) is

$$dx = R(1 - e^{-x})dt + C^{1/2}dW, \tag{1.2}$$

where payments $P(t)=P_0$ are paid continuously. Another mode to pay BSEs is to pay them in a lump sum once a year, or discretely. We consider both modes of payment starting with the continuous one. Because stochastic equation (1.2) with initial condition ($5a$.1) has no exact solution, we try to understand the process behavior in the open space ($-\infty < x < \infty$) qualitatively using the Brownian model. Suppose that at $t = 0$ we have an ensemble of Brownian particles whose initial locations x_0 have a normal distribution $V_0(x; H_0, \sigma_0^2)$ with mean H_0 and standard deviation σ_0, and one part of the ensemble is over line $x = 0$, while the other is under it ($H_0 > 0$, the *x*-axis shows up). In these conditions the line $x = 0$ is the line of a balance between the firm's mean year returns $R\langle X_0\rangle$ and the year BSE payments *P*.

It follows from Eq. (1.2) that the drift rate depends on *x*. At $x = 0$ the drift is zero; this line is the line of unstable equilibrium for the process *x*(*t*). For $x > 0$, the particles drift up, the faster the greater *x* (a positive repulsion from the line $x = 0$), with its drift rate bounded from above by *R*. For $x < 0$, the particles drift down the faster the greater $|x|$, with no limit for the drift rate at all. The two repulsion forces decrease a concentration *n*(*x*) of particles around the line $x = 0$, and the concentration of particles under the line drops faster and lower than the concentration over the line.

Thus, a diffusion force appears proportional to the concentration gradient $-dn/dx$ acting across the line $x = 0$ and driving the particles located over the line *against* the repulsing force to the line $x = 0$ and farther down. Below the line $x = 0$, the diffusion force fades fast because $-dn/dx$ tends to zero, but the negative repulsion force carries the particles farther down. The decreasing concentration of particles in a thin layer over line $x = 0$ makes the particles in the next layer over the first, so far traveling up, to stop and then move downwards. The line $x = EQ(t)$, where particles make this U-turn, floats up and up until all particles in the ensemble find themselves under the line and moving down. Over the line $x = EQ(t)$, the particles move up, under it, they move down, on the line, their drift rate is zero. The diffusion force acting between the lines $x = EQ(t)$ and $x = 0$ transports the particles across the line $x = 0$, then the negative repulsion force drives them to negative infinity. The initially normal distribution of Brownian particles by and by turns into a leptokurtic and negatively skewed x-distribution. At that, the higher the ensemble location over the line $x = 0$, the more time the ensemble deformation takes, the lesser the distribution skewness for fixed time intervals (not too long). The distribution rise in the early stage of its development can slow down to a certain extent the diffusion mass transfer across the line $x = 0$ impeding the skewness growth. Nevertheless, the permanently growing distribution skewness makes the distribution mean *a concave-down function of time* for *any* initial location of the distribution center H_0. Vice versa, the closer the mean of the initial ensemble to the zero line, the faster runs the distortion of the initially normal distribution. The described ensemble evolution explains the space-time development of the x-distribution.

It is clear that for a concave-down function of mean returns $H(t)$, the *mean year returns* are a non-increasing monotone function of time whose values keep constant for some time, then gradually decline to zero and further down to negative. This behavior of the firm's mean year returns make its stock price to decline. In these conditions, no time-sequence of the firm's stock prices can be a martingale, or the time-independent martingale (risk-neutral) measure does not exist. Later we shall see that for firms with high H_0 values, mean year returns remain about constant for their specific times $t \leq T_{NA}$, and the martingale measure exists in that time intervals $(0, T_{NA})$. At that, the lesser H_0, the shorter is the interval where the martingale measure exists, if any.

Now let us consider the case of a firm with debt X_D who pays its BSEs in a lump sum at the end of a year. The firm log-value distribution evolves in a *semi-open space* constrained from below with an absorbing boundary – the *default line* ($z = ln\,(X / X_D)$, $0 \leq z < \infty$). The exogenous boundary line first appears in (Black & Cox 1976) for GBM without payments. When firm's payments are taken into account, the part the default line plays in the distribution formation becomes even more significant.

Between the payments, the equation for log-value z is

$$dz = Rdt + C^{1/2}\delta W, \tag{2.2}$$

and the probability distribution $Y(z, t)$ is described by the equations

$$Y_t + RY_z - 0.5CY_{zz} = 0, \quad R = \alpha - C/2\ , \tag{3.2}$$

$$Y(z, 0) = Y_0(z;\ H_0, \sigma_0^2), \tag{4.2}$$

$$Y(0, t) = 0. \tag{5.2}$$

The function $Y_0(z;\ H_0, \sigma_0^2)$ is a normal function with the mean $H_0 > 0$ and variance σ_0^2. The firm pays its BSEs P_k, $k = 1, 2, 3, \ldots$, at the end of each kth year. We shall discuss the part which the payments play in problem (3.2) – (5.2) interpreting it as a problem of finding a temperature distribution for a temperature (heat) wave slowly ($d\sigma/dt > R$) propagating along a thin metal rod of infinite length. The left end of the rod at $z = 0$ is in a heat contact with a vessel of infinite heat capacity at zero temperature (so, the rod temperature at $z = 0$ is always zero, Eq. (5.2). Each point of the rod represents a firm log-value, and the temperature distribution represents the probability distribution for the firm log-value. At time $t = 0$, a temperature packet of a profile (4.2) appears at the rod and starts its propagation to the right at the rate R. If $H_0 \gg \sigma_0$, the temperature packet travels right with the drift rate R: $\widehat{H}(t) = H_0 + Rt$, spreading with a rate $C^{1/2}$: $\hat{\sigma}(t) = \sqrt{\sigma_0^2 + Ct}$. Here the mean $\widehat{H}(t)$ and variance $\widehat{V}ar(t) \equiv \hat{\sigma}^2(t)$ are

$$\widehat{H}(t) = \int_0^\infty zY(z, t)dz \tag{6.2}$$

$$\hat{V}ar(t) = \int_0^\infty \left(z - \hat{H}\right)^2 Y(z,t)dz \tag{7.2}$$

When the left edge of the packet reaches $z = 0$, a heat flow appears from the rod to the vessel proportional to a temperature gradient at the boundary, and the left end of the rod starts cooling. The left-end temperature decrease causes the temperature gradient along the rod Y_z and a heat flow $F(z, t)$ inside the packet (γ is a coefficient of proportionality):

$$F(z,t) = -\gamma Y_z(z,t),$$

at first, affecting mainly the left part of the packet close to the boundary. Another consequence of the heat flow is temperature redistribution from the right hotter area to the left cooler area inside the packet. This redistribution decreases the effective drift rate slowing down the mean rate of packet propagation along the rod. At the same time, the more heat is redistributed from right to left, the faster runs cooling at $z = 0$. At the end of the first year, the firm pays its BSEs, P_1; in the suggested interpretation, that means an instant shift of the temperature distribution to the left, increasing the temperature gradient at $z = 0$ and speeding up the rod cooling.

The described process repeats itself each year: the heat packet propagates to the right, but with a slower effective drift rate caused by the heat redistribution inside the packet from right to left. At the end of the kth year, $k = 2, 3, \ldots$, the lump sum of P_k is paid instantly; for the heat propagation problem, it means shifting the temperature distribution to the left; thus, speeding up the rod cooling.

As a result, after some time all the packet heat will be absorbed by the vessel leaving the rod with zero temperature in it. The distribution mean $\hat{H}(t)$ occurs to be a piecemeal function with jumps down resulting from instant left shifts in the distribution. Between the jumps, the function $\hat{H}(t)$ consists of segments of a concave-down function of time: at first it grows over H_0 due to the packet propagation to the right, then this growth slows down as the more and more heat inside the packet is redistributed from the right hot part of the temperature distribution to the left cooler part and $\hat{H}(t)$ achieves its maximum; after that, the function $\hat{H}(t)$ decreases to zero as the packet heat is more and more absorbed in the vessel. As an amount of heat in the rod decreases, the length of jumps down by and by declines to zero.

It is clear from this line of reasoning that the introduction of the absorbing screen (a default line) into the open-space problem (3.2) - (4.2) makes the log-value mean (the mean returns) a concave-down function of time. The firm's payments speed up the development of the concave-down mean and introduce additional leaps down into that mean. So, the papers (Giesecke & Goldberg 2008, Strebulaev 2007) are inconsistent with their goals to estimate the firm's default probability because the authors use the martingale technique *and* the default line (the absorbing screen). The default line makes the firm's mean returns a concave-down function of time. As one knows, the martingale technique needs the linear mean value $H(t) = H_0 + Rt$, which is correct for safe, prosperous firms only. When the firm meets serious credit risks, its mean year returns begin to decline, and the martingale technique is not applicable anymore.

The variance $\hat{V}ar(t)$ is also a piecemeal function with jumps at times of BSE payments. However, the pattern of variance jumps is more complicated compared to that of the $\hat{H}(t)$ jumps. At first, the variance grows over the line $\hat{\sigma}^2(t) = \sigma_0^2 + Ct$ due to the packet spread caused by thermo-diffusion and the packet deformation caused by inner heat flows; during this stage, the firm's payments cause the variance to jump up. By and by the variance achieves its maximum when the expanding trend is compensated by the shrinking trend caused by a general loss of heat at the boundary. As the variance achieves its maximum, the jumps up become shorter and shorter, and after the maximum, they become negative. When the shrinking trend prevails, the variance falls down to zero as the packet's heat is absorbed in the vessel. During this stage, the lengths of jumps down tend to zero. Between the jumps, the variance consists of segments of a smooth function; at first this function is a concave-up function, but as the variance approaches its maximum, the function becomes a concave-down function and it remains in that state until the variance comes to zero.

Because of the internal temperature redistribution, the originally symmetric packet loses its symmetry. At the early stages of the packet development, it acquires negative skewness $\hat{S}(t)$, but later the packet skewness assumes small positive values, and then it finally fades to zero. As in the cases of the mean or variance, the firm's payments at the end of the kth year show themselves in skewness jumps.

$$\hat{S}(t) = \int_0^\infty \left(z - \hat{H}\right)^3 Y(z,t)dz \tag{8.2}$$

Within any kth year, k = 1, 2, …, the *effective rate of year returns* does not remain constant but is a decreasing monotone function of time. It makes the firm's stock price to decrease over time, and, consequently, the time-sequence of stock prices can never be a martingale, or the time-independent martingale measure does not exist. However, the growth of the mean value caused by the initial positive drift can counteract to a certain extent the effect of the firm value redistribution to $z = 0$ inside the distribution $Y(z, t)$ slowing down the development of skewness. For sufficiently high H_0 values, mean year returns can remain approximately constant in a time interval $(0, T_{NA})$ individual for each firm, and the martingale measure exists in that interval. Vice versa, the closer H_0 to $z = 0$, the faster runs the distortion of the initially normal distribution, the shorter is the time interval where the martingale measure exists, if any.

According to the *First Fundamental Theorem of Asset Pricing* (Shiryaev 1998, p. 259), a (B, S)-market determined in a filtered probability space $(\Omega, F, (F_n), P)$ consists of a bank account $B = (B_n)$, $B_n > 0$ and a finite number d of assets $S = (S^1, S^2, \dots, S^d)$, $S^i = (S^i_n)$. The market operates at time moments $n = 0, 1, \dots, N$, $F_0 = \{\emptyset, F\}$, $F_N = \{F\}$.

The (B, S)-market is a no-arbitrage pricing market if and only if there is a martingale (risk-neutral) measure $P^{\sim}$ equivalent to the P-measure, and a d-dimensional calibrated sequence

$$\frac{S}{B} = \left(\frac{S_n}{B_n}\right), \quad S_n = (S_n^1, S_n^2, \dots, S_n^d) \tag{9.2}$$

is a $P^{\sim}$-martingale, that is, for any $i = 1, 2, \dots, d$ and $n = 0, 1, \dots, N$, one has

$$E_{P^{\sim}} \left|\frac{S_n}{B_n}\right| < \infty, \quad E_{P^{\sim}} \left(\frac{S_n}{B_n}\middle| F_{n-1}\right) = \frac{S_{n-1}^i}{B_{n-1}^i}. \tag{10.2}$$

One can find the Fundamental Theorem of Asset Pricing in a bit different wording in an excellent textbook (Financial Economics 1998, p. 525). A crucial condition for getting (10.2) is a self-financing portfolio, which means that all portfolio incomes go for further investments only; there is no other outflow from this asset portfolio (Harrison & Krebs 1979). Using the martingale terminology, one can say that the calibrated stock price sequence (S_n/B_n) makes a *local martingale* for the firms with payments.

Using this theorem, one can conclude that the market for which there is no risk-neutral measure is not the no-arbitrage pricing market. This explanation shows that neither the model with discrete-time payments nor the model with continuous-time payments supports the ideas of risk-neutral probabilities and no-arbitrage pricing markets in general. The no-arbitrage pricing principle holds only for individual stocks and in their specific time intervals $0 \leq t \leq T_{NA}$.

General understanding of the processes (5.1) and (2.2) helps make conclusions on the intensity of default probability (*IPD*) development. Because the process drives particles to negative infinity and the faster, the deeper their locations under the line $x = 0$, the intensity of default probability *IPD* grows with acceleration over time. For the firm with discrete payments, the function $IPD(t)$ is a piecemeal function with jumps up at the moments of BSE payments; between the jumps the function consists of pieces of a continuous concave-up function. The default probability *PRD* as an integral of *IPD* over some time interval is also an increasing monotone function of time.

We solve the problem (6.1), (7.1), (10.1) with $\pi(t) \equiv 1$ numerically, estimating the mean returns $H(t)$, variance $VAR(t)$, skewness $S(t)$, and the intensity of default probability $IPD(t)$ for x-distribution. We trace down a dependence of the default probability on factors of the firm and its business environment. Knowing the risk level threatening a firm is important not only for banks issuing commercial credits to the firm, or for the companies insuring those credits, but also for the firm's management when planning long-term business operations. At this stage, we do not include taxes or dividends; we suppose that all *perfect market* assumptions hold. Figures 1 – 4 present examples of modeling of the distribution $V(x, t)$, and its statistical moments $H(t)$, $VAR(t)$, $S(t)$, and Fig. 5, 5a show the intensity of default probability $IPD(t)$ for different initial conditions. Model parameters are $R = 0.10$, $\sigma_0^2 = 0.03$, $C = 0.008$ (yr^{-1}), $T = 10$ (yr), $DL = 0$.

Table 1. Relation between the initial value H_0 and a share of BSEs in mean year returns

H_0	$\frac{P}{R\langle X_0\rangle}$	H_0	$\frac{P}{R\langle X_0\rangle}$	H_0	$\frac{P}{R\langle X_0\rangle}$	H_0	$\frac{P}{R\langle X_0\rangle}$
4.0	0.0183	1.4	0.248	0.9	0.407	0.4	0.670
3.0	0.050	1.3	0.273	0.8	0.449	0.3	0.741
2.5	0.082	1.2	0.301	0.7	0.497	0.2	0.819
2.0	0.136	1.1	0.333	0.6	0.549	0.1	0.905
1.5	0.223	1.0	0.368	0.5	0.607	0.	1.

Fig. 1 shows the difference $H(t) - H_0$ as a function of time and H_0. The choice of H_0 values is explained in *Table 1*, illustrating a relation between the parameter H_0 and the BSE share in the mean year returns $P_0/(R\langle X_0\rangle)$. The first two entries in the table (H_0 = 4.0 and 3.0) showing too low BSE shares are included here to demonstrate that the difference $H(t) - H_0$ for x-distribution tends asymptotically to $H(t) - H_0 = Rt$, specific for GBM as H_0 tends to infinity.

Fig. 1 and Fig. 1a supporting the inference of our qualitative analysis demonstrates that all lines H(t) fall apart into two classes: the class of lines first rising, then falling, and the class of lines declining from the start. The second class is of no practical interest and not considered here. This division is mainly controlled by H0-parameter; other problem parameters (R, σ_0^2 , and C) contribute to a lesser degree. The critical H0 for the chosen problem parameters is about one. Among the rising and falling lines, there are lines whose rise takes a long time (decades, lines 1 – 6). One can consider such lines as semi-steady ones in not too long time intervals. Fig. 1 shows that a slope of each H(t) – H0 line declines due to losses at the default line (see Eq. (9.2)) from R specific for the asymptotic GBM-distribution as H0 descends from high to low values. The lines with H0 in the interval (1.10, 1.20) are the lines of stagnation for whom H(t) varies within $\pm 0.1H_0$ at the interval of ten years. Fig. 1 proves that the GBM-distribution mean is a poor approximation for the realistic EMM lines H(t). It is important to understand that all H(t) – H0 curves are concave down (like lines 7 and 8), and sooner or later these lines shall pass their maximum and decline.

The approximately straight rise of H(t) for sufficiently high H0 (lines 3 – 6) providing for the almost constant mean year returns says that the no-arbitrage pricing principle is valid for the assets (stocks or bonds) issued by those firms within the considered time intervals. However, this principle does not hold at the entire market, but it is a characteristic of an individual asset and the firm standing behind it. Fig. 1a shows clearly the decrease in mean year returns for the firms with H0 = 1.20, 1.15, and 1.10. Mark that line 1 in Fig. 1a (H0 = 1.20) remains straight for t ≤ TNA = 8 years, while line 2 (H0 = 1.15) remains straight only for t ≤ TNA = 4 years. Line 3 is never straight. Stock prices of the firm 1 (H0 = 1.20) can be constant for t ≤ 8 years, stock prices of the firm 2 (H0 = 1.15) are constant for t ≤ 4 years, and stock prices of the firm 3 (H0 = 1.10) are never constant. So, the no-arbitrage pricing principle could be effective for the stock 1 for 8 years, for the stock 2 for 4 years, and it is never effective for the stock 3. However, growing volatility introduces its corrections to the intervals where the stock prices remain constant.

Of course, real TNA estimates are determined by the $\widehat{H}(t)$ solution:

$$\widehat{H}(t) = \int_{DL}^{\infty} x\widehat{V}(x,t)dx,$$

here $\widehat{V}(x,t)$ is the solution of the boundary problem (6.1) – (8.1). Real T_{NA} values calculated with $\widehat{H}(t)$ are shorter than the values estimated above.

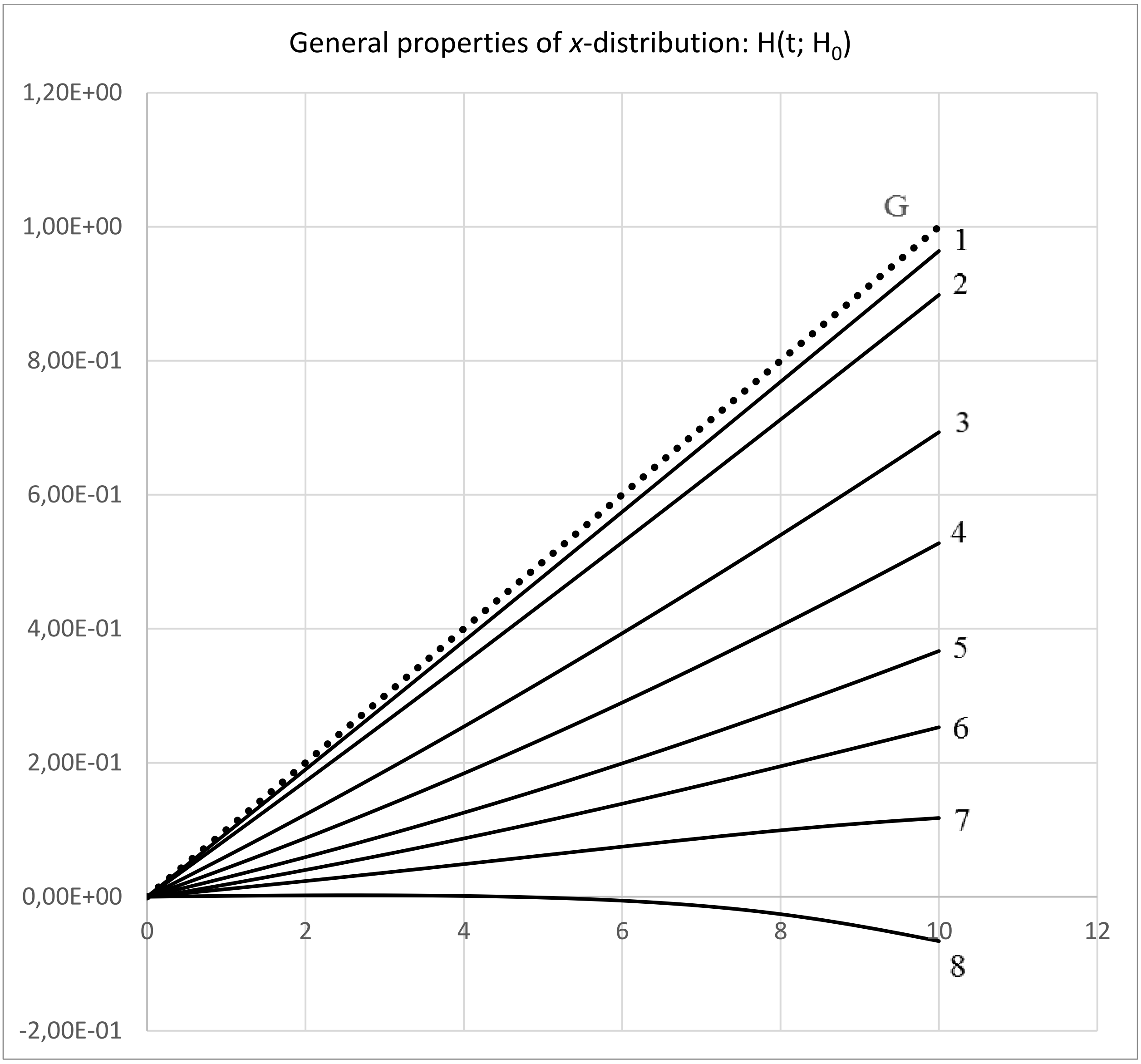

Figure 1. Log-value mean $H(t;\, H_0) - H_0$ as a function of time (years) and initial H_0 values: $H_0 = 4.0$ (1), 3.0 (2), 2.0 (3), 1.6 (4), 1.4 (5), 1.3 (6), 1.2 (7), 1.1 (8). The straight dot line (G) shows a drift of the asymptotic GBM solution $\Delta H = Rt$.

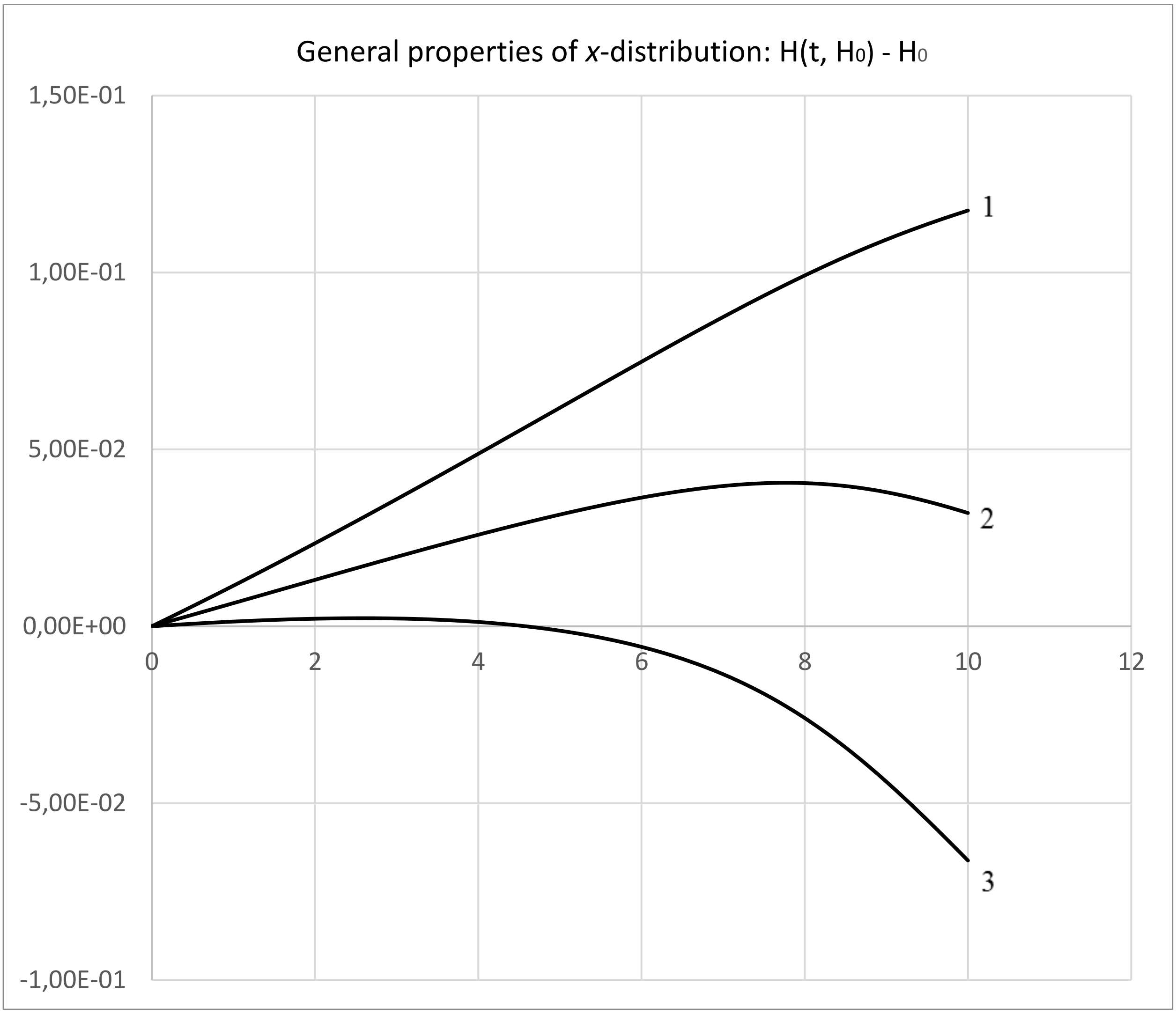

Figure 1a. Log-value mean $H(t, H_0) - H_0$ as a function of time (years) and initial H_0 values. $H_0 = 1.20$ (1), 1.15 (2), 1.10 (3).

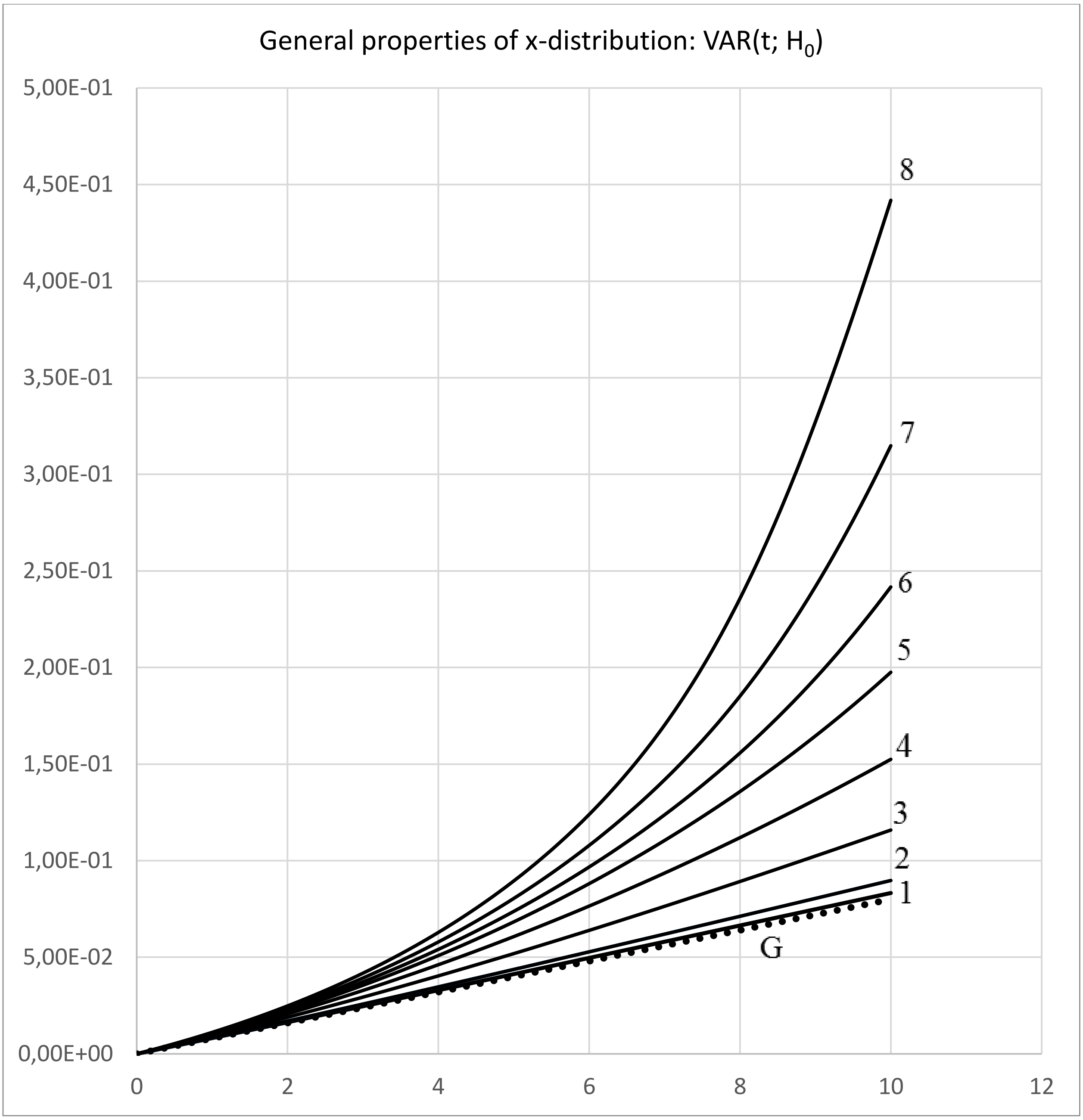

Figure 2. The variance $VAR(t;\, H_0) - VAR_0$ as a function of time (years) and initial values H_0 = 4.0 (1), 3.0 (2), 2.0 (3), 1.6 (4), 1.4 (5), 1.3 (6), 1.2(7), 1.1 (8). The straight dot line (G) shows the spread of the asymptotic GBM solution $\Delta VAR = Ct$.

The x-variance $VAR(t)$ (Fig. 2) demonstrates the pattern also supporting our qualitative analysis. When H_0 descends from high to low values, the variance grows fast, starting from the Ct-line specific for the GBM-distribution. For small t near the start where the x-distribution skewness is still low, the variance is close to its GBM-approximation. When x-distribution gains material skewness, its variance significantly exceeds the GBM-variance. Variance values for the cases with low H_0-parameters in the end of the ten-year period show violent fluctuations: for H_0 = 1.1, the standard deviation increases from σ_0 = 0.17 to σ = 0.69 (VAR = 0.47), while for GBM-distribution the standard deviation is half as much (VAR = 0.11, σ = 0.33). The excess of x-variance over GBM-variance is due to the distribution deformation. On the contrary to the GBM-variance having constant volatility $C^{1/2}$, the x-variance has an effective volatility $C_{eff}^{1/2}(0) = C^{1/2}$, $C_{eff}^{1/2}(t) = (dVAR/dt)^{1/2}$, which is an increasing monotone function of time. The growing volatility shortens the time intervals where the stock prices remain constant (see Fig. 1 and 1a).

There is a widely known empirical fact that debt increases the firm's asset variance. No GBM can explain it because for GBM, one has $\sigma^2(t) = \sigma_0^2 + Ct$ for both the identical levered and unlevered firms (see Eq. 3.1 and Fig.

2). To clarify this phenomenon using EMM, let us consider two identical firms: the levered one and the unlevered one. The identity means that the following characteristics of the firms are the same: mean initial assets $\langle X_0 \rangle$, the expected rates of return R, the initial variances σ_0^2 , fixed cost payments P, volatility $C^{1/2}$. The levered firm has a debt X_D in its assets, and, therefore, the business securing payment for the levered firm makes $P_L = P + DP > P = P_{UL}$, DP is a debt payment. The initial distribution center of the unlevered firm value in the logarithmic space is $H_0^{UL} = \langle \ln{(RX_0/P)} \rangle$, and the initial distribution mean of the levered firm value is $H_0^L = \langle \ln{[RX_0/(P+DP)]} \rangle$. As one can see, the initial position of the levered firm is below the initial position of the unlevered firm, $H_0^{UL} > H_0^L$. As a consequence, the skewness of the levered firm distribution grows faster than the skewness of the unlevered firm distribution. An extra distribution distortion makes a higher contribution to the firm's variance making the difference between the firms' asset variances continuously grow over time (Fig. 2).

We see a similar pattern in the development of x-distribution skewness (Fig. 3 and 3a). From small negative values specific to high H_0 (4.0 or higher), it declines fast, taking on large negative values, about – 0.5 (H_0 = 1.1). For the standard skewness S_{st}:

$$S_{st} = S/VAR^{3/2}$$

and $H(t = 10, H_0 = 1.1)$, one has S_{st} = – 1.33. The S&P500 Index's average value for 1970 – 2000 is S_{st} = –1.73 (Kou, 2007). Typical examples of x-distribution are shown in Fig. 4, where one can observe the development of long negative tails.

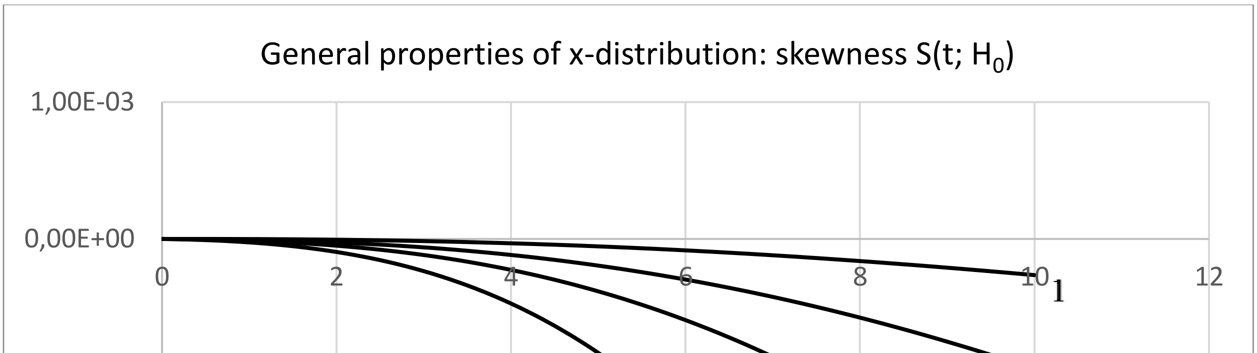

Figure 3. Development of the x-distribution skewness S as a function of time (years) and values H_0 = 4.0 (1), 3.0 (2), 2.0 (3), 1.6 (4).

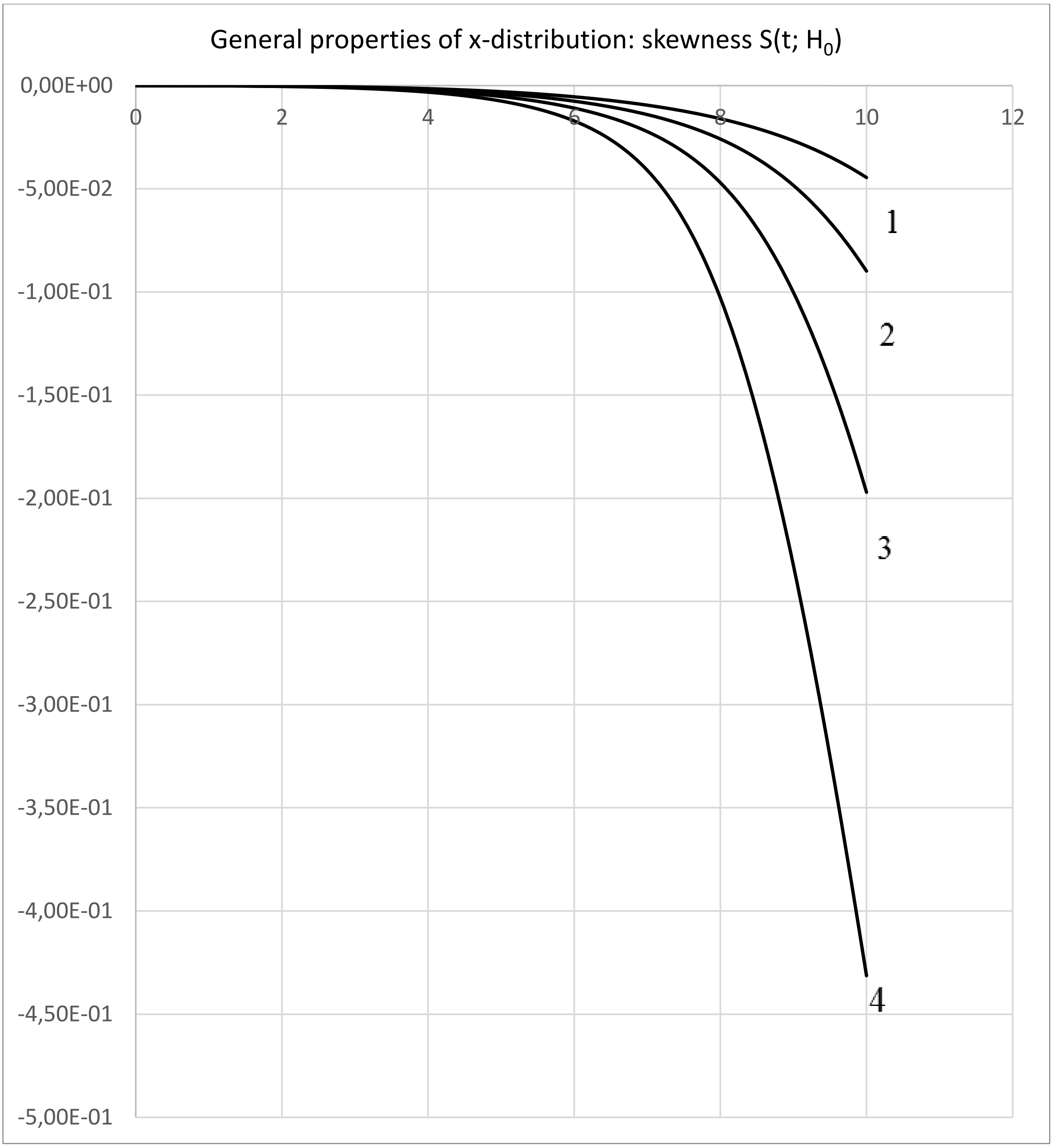

Figure 3a. Development of the *x*-distribution skewness *S* as a function of time (years) and H_0 = 1.4 (1), 1.3 (2), 1.2 (3), 1.1 (4).

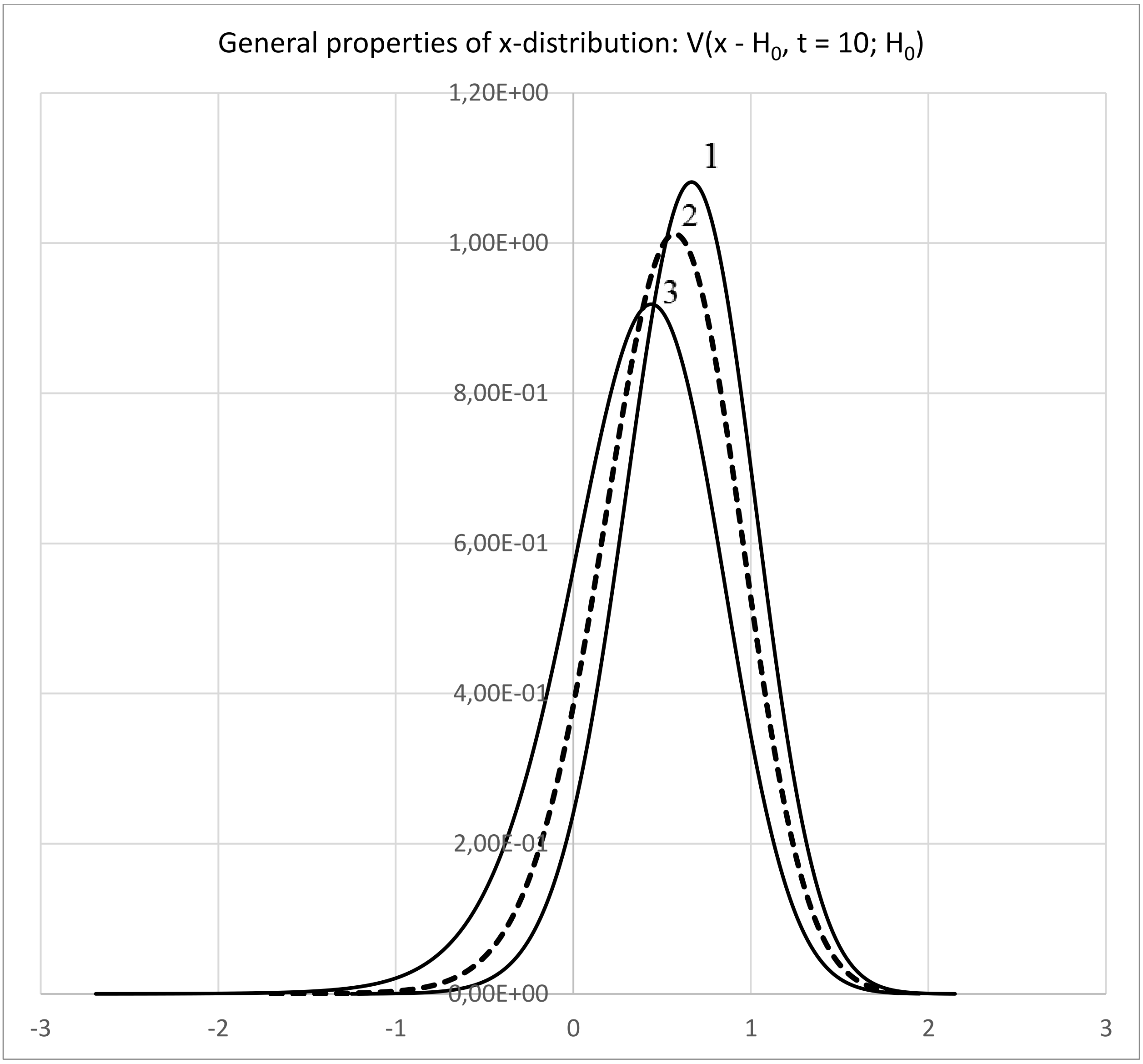

Figure 4. The x-distribution $V(x - H_0, t = 10)$ for H_0 = 1.8 (1), 1.6 (2), 1.4 (3). Observe development of a negative tail and increasing distribution skewness.

Fig. 5 and 5*a* show how the intensity of default probability *IPD*(t, H_0) depends on time and the initial mean H_0. Mark that development of *IPD*(t, H_0) is very inertial: for a significant part of the ten-year interval *IPD* remains low, rising to noticeable values in the second half of the graph. Fig. 5 and 5*a* demonstrate a fast *IPD* rise when H_0 decreases. This behavior of the *IPD*(t) function can suggest a false feeling of safety to the firm's management using one-year horizon models like Moody's KMV (Bohn, 2006) for estimating firm's credit risks. Such models warn about the oncoming crisis too late when a significant time, so necessary for improving the firm's position, is already wasted.

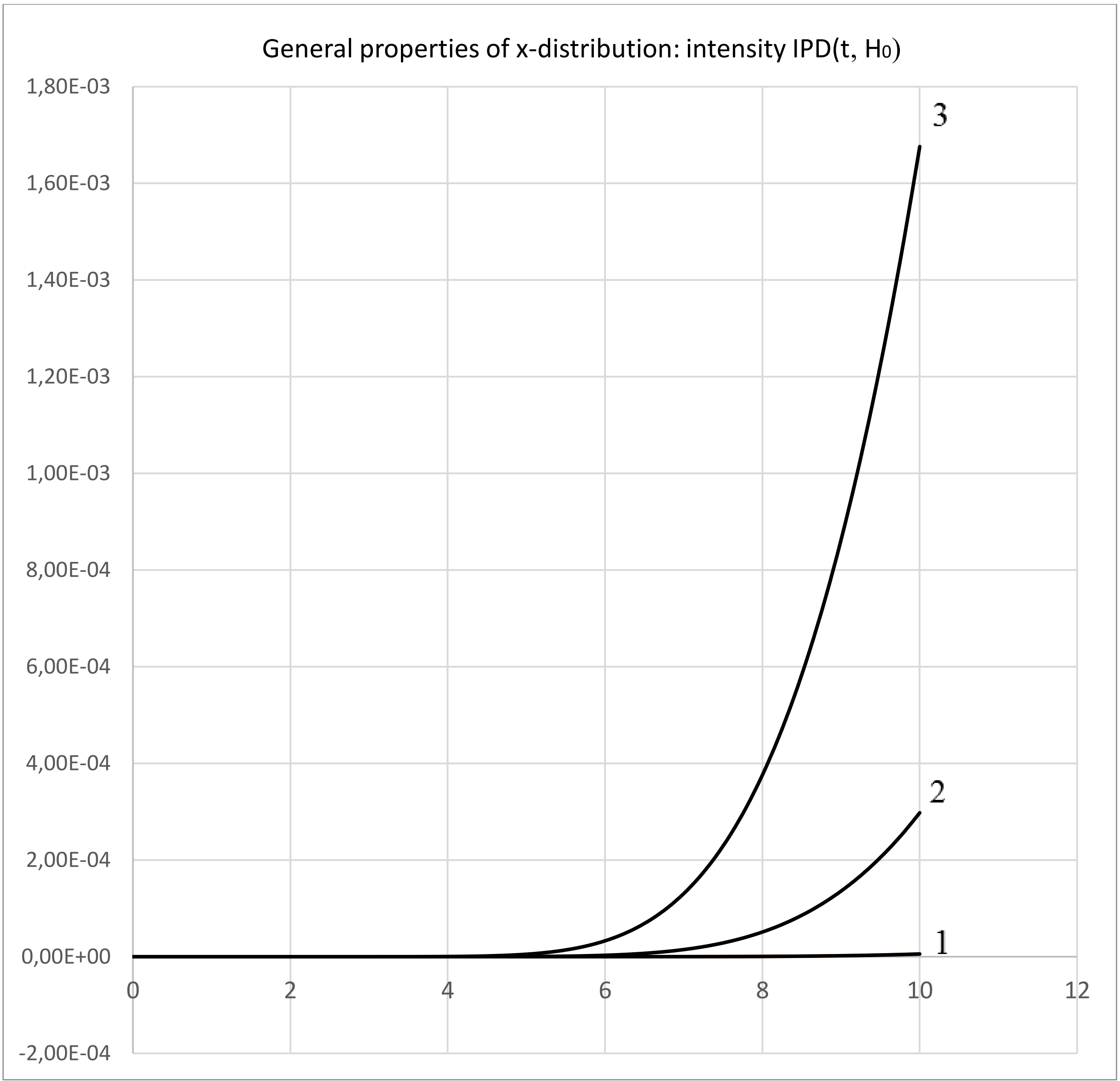

Figure 5. The intensity of default probability *IPD*(t, H_0) as a function of time (years) and the initial values H_0 = 1.80 (1), 1.6 (2), 1.50 (3).

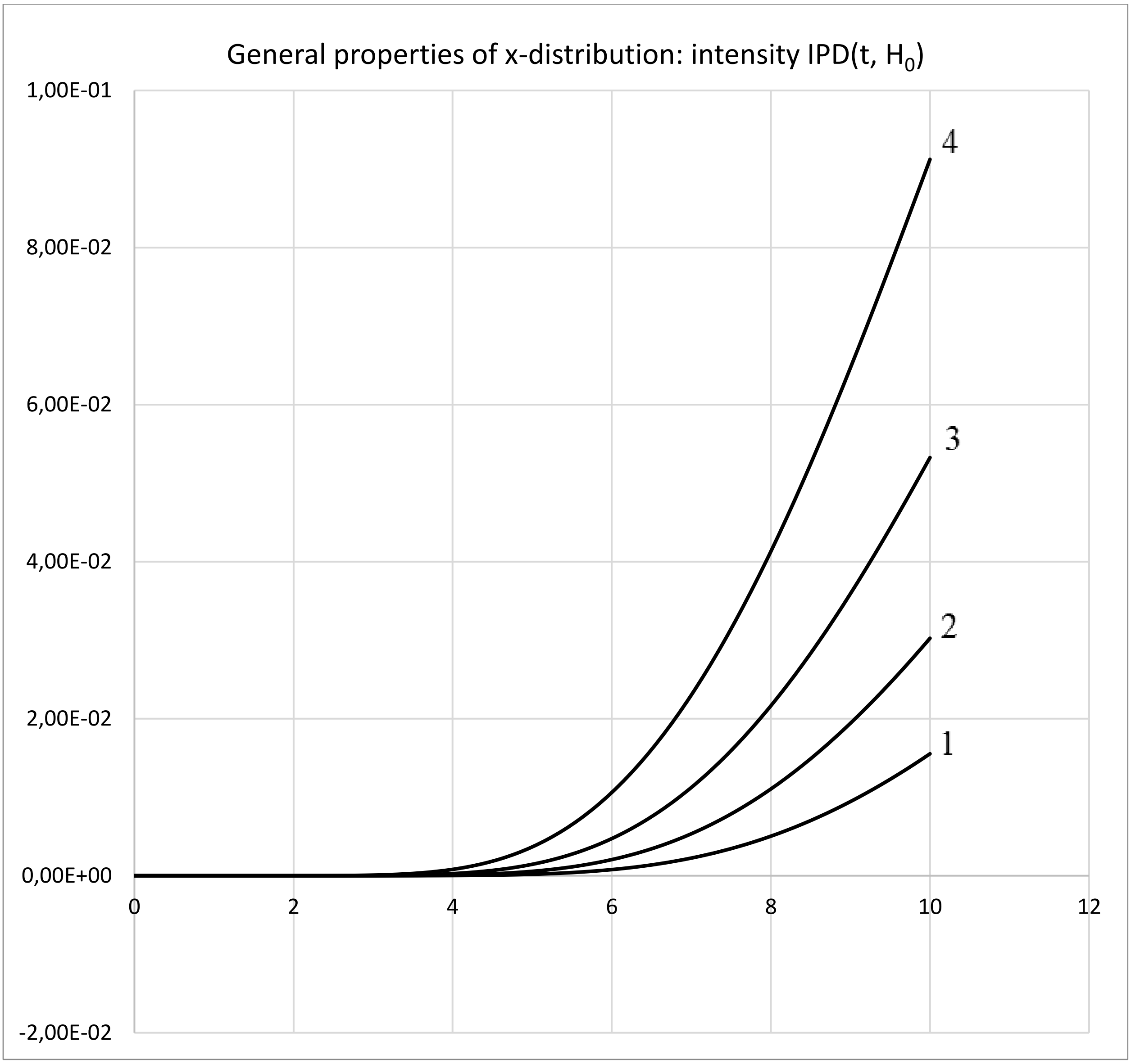

Figure 5a. The intensity of default probability *IPD*(t, H_0) as a function of time (years) and initial values H_0 = 1.35 (1), 1.30 (2), 1.25 (3), 1.20 (4).

The no-arbitrage pricing principle effectiveness for the whole market follows from the GBM neglecting the firm's payments. For the firms paying their BSEs, this principle holds for an individual stock, and only at the time interval $t \leq T_{NA}^{i}$ determined by the ith stock traded at the market. Therefore, the no-arbitrage pricing principle is never effective for the whole market because the firms start their businesses at different times independently of each other. However, for short-term operations when the traders buy firms' stocks and soon resell them trying to profit from the price difference, the market is always no-arbitraging and "fair". For short-term deals, one can neglect the firm's payments, and calibrated prices of any stock make a martingale. Considering long-term investments ($T_{inv} \approx T_{NA}^{i}$) when mean year returns begin to decline, an investor must be very cautious because the stock price martingale characteristic and no-arbitrage pricing principle become invalid. Increases in debts, taxes, or inflation can significantly decrease the firm mean year returns inflicting losses to the long-term investor.

The principal difference between the lognormal GBM-distribution and the skewed EMM-distribution is the following. The GBM-firm remains "ever young", keeping time-invariant mean year returns R and volatility $C^{1/2}$; it dies (defaults) among full prosperity in an accident. A diffusion walk of a stochastic firm value is symmetric to both sides. The probability that a diffusion process starting at time $t = 0$ from the point M on a plane (x, t) shall cross an arbitrary straight line $x = a$ in that plane within a finite time $T(M, a)$ equals unit almost for sure: $P(T(M, a) < \infty) = 1$, a. s. (Shiryaev 1998, pp. 302-303). If the line $x = a$ is the default line, then the firm's longevity is finite almost for sure. From the optimistic point of view, that means that the firm's longevity is infinite for a set of firms of a null measure: $T(M, a) = \infty$, and, theoretically, some firms can exist forever! On the contrary to the ever-young GBM-firm, the EMM-firm gradualy "grows old": its effective mean year returns $R_{eff}(t)$, $0 < R_{eff}(0) < R$, is a non-increasing monotone function of time, and its effective volatility $C_{eff}^{1/2}(0) = C^{1/2}$, $C_{eff}^{1/2}(t) = (dVAR/dt)^{1/2}$, and skewness are increasing monotone functions of time. No later than at time t_{Max} :

$$t_{Max} = \min\left(t_R, t_C;\ t_R\colon R_{eff}(t_R) = 0;\ t_C\colon C_{eff}(t_C) = C_{cr}\right),$$

the firm's investors will lose their interest in the firm, get rid of its stocks, and the firm will soon default (here $C_{cr}^{1/2}$ is critical volatility at which investors recognize the risk as unacceptable). A diffusion walk of the firm value is now faster ($VAR_{EMM}(t) \geq VAR_{GBM}(t)$) and asymmetric: a negative shift in the firm value is more likely than a positive shift (the EMM-firm meets more black swans than white ones). This asymmetry increases the default probability of the EMM-firm in comparison to the identical GBM-firm. The EMM-firm defaults accidentally any time before or about the time t_{Max}. GBM-estimations of the firm's longevity are overoptimistic and therefore misleading.

On Modigliani-Miller Propositions

Now our objective is to show that MM Propositions (Modigliani & Miller 1958, 1961, 1963) follow from the perfect market conditions joined with an implicit use of GBM for description of the mean firm value.

The idea of proof of MM Propositions consists in the development of an analog of the Marshallian industry for firms' cash flows with the following application of the one price principle to the market of perfect substitutes. Modigliani and Miller (1958) consider firms at the perfect market described with the following assumptions:

A. The firm value is determined only by the mean cash flow generated by the firm;
B. All investors have complete information about firms' cash flows; thus, the investors have homogenous expectations on corporate cash flows and their riskiness;
C. There is an "atomistic" competition and no market friction of any kind. That implies among other things that at the market of corporate stocks and bonds (a) there are no agency costs, (b) bankruptcy entails no liquidation costs, and (c) all investors both individuals and institutions can borrow at the same rate as corporations;
D. The debt of firms and investors is riskless, so the interest rate of all debts is the risk-free rate for all possible amounts of debt;
E. There are no corporate or personal taxes.

The authors argue that all "firms can be divided into "equivalent return" classes such that the return on the shares issued by any firm in any given class is proportional to (and hence perfectly correlated with) the return on shares issued by any other firm in the same class" (Modigliani & Miller, 1958, p. 266). They insist that "all relevant properties of a share are uniquely characterized by specifying (1) the class to which it belongs and (2) its expected return", and by that they create "an analog to the industry in which it is the commodity produced by the firms is taken as homogenous" (Ibid., p. 266). In a later paper, Miller confirms that for the "equivalent return class" (here he calls it the "risk class") "the uncertain, underlying future cash flow streams of the individual firms within each class could be assumed perfectly correlated, and hence perfect substitutes", and further: "at the practical level, the risk class could be identified with Marshallian industries" (Miller 1988, p. 103). As one can see, finding two firms with perfectly correlated random cash flows is a rare event implying that only a few firms, if any, can fall in each risk class. Ross (1988) is the first who argues that Miller's interpretation of the risk class is too narrow. Trying to improve it, he says (Ross, 1988, p. 130) that "two cash flow streams need not to be perfectly correlated to be […] placed in the same risk class. If the implication of being in the same risk class is that securities have the same cost of capital, then, since the cost of capital depends on the systematic risk of a company, we have seen that two firms with completely uncorrelated risks, risks that are independent of the major systematic risks in the economy, can be placed in the same risk class." However, the implication that the levered and unlevered firms have the same cost of capital follows from MM Propositions; thus, this attempt of extension of the risk class has failed.

By the method of proof, Modigliani and Miller claim that their Propositions are invariant to the initial distribution of the firm value and, therefore, the Propositions are universal at the equilibrium perfect market. Now we know that the firm value distribution must meet Eq. (5.1). The assumed ability of the two identical levered and unlevered firms to get into and stay long enough in the same risk class (or equivalent return class) for any asset structure is effectively the assumption that the firm value distributions are *lognormal*. (The levered firm is identical to the unlevered one in any respect, but the structure of its assets.) Really, *the mean returns of the levered firm must be equal the mean returns of the unlevered firm at all times*: $H_L(t) = H_{UL}(t),\ t \geq 0$. It is possible for the lognormal distribution only, for which one has $H(t) = H_0 + Rt$, Eq. (3.1). From the assumption of the firms' identity and the distribution lognormality, one has the conditions $H_0^L = H_0^{UL} = H_0,\ R_L = R_{UL} = R$. Therefore, the assumption that the levered and unlevered firms have the same mean returns implies that the both firms have lognormally distributed values. The lognormal distribution is a solution of GBM without payments ($P = 0$), or using proportional BSE payments $P = \delta X$, δ is constant. We consider these two cases separately.

The condition $P = 0$ means that neither the levered firm nor the unlevered firm pays any BSEs, and, in particular, dividend *DIV* and debt *DP*, see Eq. (2.1). Because there is no dividend payment for both firms ($DIV = 0$), the dividend policy makes no effect on the firm value (MMP2, 1961). Because there is no debt payment ($DP = 0$), the asset structure of the levered firm does not affect the firm value (MMP1, 1958). Because the "levered" firm does not pay for its debt but *presumably enjoys a tax shield*, its mean after-tax value is higher than the mean after-tax value of the unlevered firm by the present value of the tax shield (MMP3, 1963). However, the following argument shows that MMP3 is a logical error. Because the levered firm does not pay for its debt, it is indistinguishable from the identical unlevered firm and, therefore, *its tax shield must be zero*! The revised MMP3 must run as: in the presence of corporate taxes, the after-tax value of the levered firm is equal to the after-tax value of the unlevered firm. As we show below, this is the maximum effect of the tax shield on the levered firm value; for all other cases, the contribution of debt and the tax shield to the firm value is *negative*. The revised effect of debt in the presence of corporate taxes makes the *traditional problem of the optimal capital structure inexistent* and *the trade-off theory false* (e. g. Kraus & Litzenberger 1973, Leland 1994a, Leland & Toft 1996, Goldstein et al. 2001). We see that MM Propositions refer to the ideal perfect market consisting of firms without payments. This abstraction is so far from reality that MM Propositions are generally misleading even at the *perfect market* made of firms paying their BSEs. However, for short-term deals ($t << 1$), when the mean value of the firm remains about constant, and one can neglect payments and use GBM, MMP1 is true.

Now let us consider a GBM-firm with BSE payments proportional to the firm value: $P = DIV + DP$ and $P = \delta X$, δ is constant, described by Eq. (2.1), (3.1). Introducing a new variable $z = \ln X$, we come to the equation

$$dz = \lambda dt + C^{1/2} dW, \tag{1.3}$$

$$\lambda = R - \delta,\ P = \delta X.$$

First, we analyze the effect of a dividend policy on the mean returns. Suppose that there are two identical unlevered firms with different dividend policies $P = DIV_1 = \delta_1 X$ and $P = DIV_2 = \delta_2 X$, $\delta_2 > \delta_1$ (in GBM, $P = \delta X$, the dividend policy is reduced to a choice of the constant δ). Then mean log-values for these firms are

$$H_1(t) = H_0 + (R - \delta_1)t,\quad H_2(t) = H_0 + (R - \delta_2)t, \text{ and } H_1(t) > H_2(t). \tag{2.3}$$

We see that dividend policy affects the mean returns and value of the firm, and MMP2 (1961) is false in GBM, $P = \delta X$.

Now we consider an unlevered firm and an identical levered firm. For the first firm $P = DIV = \delta_1 X$, for the levered firm with debt of infinite maturity $P = DIV + DP = \delta_2 X$, ($\delta_2 > \delta_1$). The means for these firms again assume the form (2.3) what implies a dependence of the mean returns and value on the asset structure, and that MMP1 (1958) is wrong in GBM, $P = \delta X$.

Now we introduce corporate taxes with the rate of *tax* into the case of the unlevered (δ_1) and identical levered (δ_2) firms ($\delta_2 > \delta_1$). At that, the year debt payment is $DP = (\delta_2 - \delta_1)X$. For log-value means of the unlevered and levered firms, one has

$$\langle z_1(t)\rangle = H_1(t) = H_0 + (R - \delta_1)t\,,\quad \langle z_2(t)\rangle = H_2(t) = H_0 + (R - \delta_2)t\,, \tag{3.3}$$

and $H_1(t) > H_2(t)$. The mean value is $\langle X(t)\rangle = \langle e^z\rangle$, and for distribution (3.1) it makes

$$\langle X(t)\rangle = \exp\{(H(t) + \sigma^2(t)/2)\}\,, \tag{4.3}$$

$$\sigma^2(t) = \sigma_0^2 + Ct \,, \tag{5.3}$$

and $\langle X_1(t)\rangle > \langle X_2(t)\rangle$. The after-tax mean value of the unlevered firm makes

$$\langle X_{AT1}(t)\rangle = tax\langle X_1(t-1)\rangle + (1- tax)\langle X_1(t)\rangle \,, \tag{6.3}$$

t – 1 denotes the year before year *t.* The after-tax mean value of the levered firm is

$$\langle X_{AT2}(t)\rangle = tax\langle X_2(t-1)\rangle + (1-tax)\langle X_2(t)\rangle + tax(\delta_2-\delta_1)(\langle X_2(t)\rangle - \langle X_2(t-1)\rangle), \tag{7.3}$$

where the last term represents the tax shield. Taking account of Eq. (3.3), (4.3) and (5.3), one can show (see Appendix) that the difference between the after-tax mean values of the unlevered $\langle X_{AT1}(t)\rangle$ and levered $\langle X_{AT2}(t)\rangle$ firms is positive:

$$\langle X_{AT1}(t)\rangle - \langle X_{AT2}(t)\rangle = tax(\langle X_1(t-1)\rangle - \langle X_2(t-1)\rangle) + (1-tax)(\langle X_1(t)\rangle - \langle X_2(t)\rangle) - \\ - tax(\delta_2-\delta_1)(\langle X_2(t)\rangle - \langle X_2(t-1)\rangle) > 0 \,. \tag{8.3}$$

This result contradicts to MMP3. So, none of the MM Propositions holds in the GBM, $P = \delta X$.

Now we show how one can achieve the above results for GBM, $P = \delta X$ through a simple line of reasoning without long calculations. Without any loss of generality, the equation describing the unlevered firm ($P = 0$) is

$$dz = R_{UL}dt + C^{1/2}dW, R_{UL} = \alpha_{UL} - C/2 \,, \tag{9.3}$$

and the equation for the identical levered firm with the BSE payment $P = \delta X$, $\delta > 0$, is

$$dz = R_L dt + C^{1/2}dW \,,\; R_L = R_{UL} - \delta. \tag{10.3}$$

We see that the equation describing the levered firm is the same as the equation for the identical unlevered firm with a correction for the expected rate of returns R. The mean returns $H(t) = H_0 + Rt$ of the unlevered firm are higher than the mean returns of the levered firm for any time $t > 0$: $H_{UL}(t) > H_L(t)$, correcting the inference of MMP1. Now let us consider two unlevered firms paying different dividends: $DIV_2 - DIV_1 = \delta X$. We see that the mean returns of the first firm are greater than the mean returns of the firm with higher dividend payments: $H_1(t) > H_2(t)$, $t > 0$, correcting the conclusion of MMP2, Eq. (2.3). Addressing the problem of values of the unlevered firm and the identical levered firm in the presence of corporate taxes, one can see that the levered firm develops as the unlevered firm with a lesser rate of expected returns. Therefore, the mean after-tax value of the unlevered firm exceeds the mean after-tax value of the levered firm:

$$\langle X_{AT}^{UL}(t)\rangle > \langle X_{AT}^{L}(t)\rangle \,,\; t > 0,$$

repeating inequality (8.3) and correcting MMP3.

In GBM, $P = \delta X$, on the contrary to MMP3, the after-tax mean value of the unlevered firm is *greater* than the after-tax mean value of the levered firm. It means that debt makes a *negative effect* on the after-tax mean value. So, the trade-off theory (e. g. Kraus & Litzenberger 1973, Frank & Goyal 2007, Strebulaev & Whited 2012) sharing the MMP3 idea of the positive effect of debt on the after-tax mean value is wrong in this model. Therefore, conclusions of the papers on the optimal capital structure based on the trade-off theory and using GBM, $P = \delta X$ are self-contradictory (e. g. Leland, 1994b; Leland & Toft, 1996; Goldstein et al., 2001, Strebulaev 2008). GBM, $P = \delta X$, *is inconsistent* with MM Propositions.

Of course, Modigliani and Miller writing their papers knew nothing about the Merton model (1974) and GBM introduced into financial engineering fifteen years later. They refer implicitly to the standard practice in financial risk estimations. This practice based on the assumption that the value distribution remains normal all the time is still in use and supported by weighty scientific sources (e. g. Brealey & Myers, 1996, Chap. 7 and other textbooks on the financial management).

It is worthy to note that in the original Merton model $(1.1a) - (1.1b)$, the unlevered firm can have the lognormal distribution, but only when it does not pay dividends to its shareholders ($P = DIV = 0$). In contrast to that, the levered firm paying no dividends still has nonzero BSE payments, $P = DP > 0$. The log-value of the levered firm starts its development from a lower initial position H_0^L than a position H_0^{UL} of the "identical" unlevered firm (see Eq. $(5a.1)$, (7.1), and Fig. 1). Its default line $DL_L = \ln(RX_D/P_L) \geq 0$, $P_L = P_{UL} + DP$, (X_D - the debt value, P_L - BSE payments of the levered firm, P_{UL} - BSE payments of the unlevered firm, DP - the debt payments) is higher than default line $DL_{UL} = 0$ of the unlevered firm (see Eq. (10.1) and comments to it). A combination of these two factors makes the log-value distribution of the levered firm negatively skewed while the distribution of the unlevered firm remains normal. The mean returns of the levered firm always run below the mean returns of the "identical" unlevered firm (Fig. 1). Hence, in EMM with continuous payments, there is *no identity* between the levered firm and the unlevered firm; they cannot fall in the same risk class even in the original Merton model. In EMM with discrete payments when the firm pays its BSEs in a lump sum once a year, characteristics of the levered firm and the unlevered one can be the same during their first year only, before the payments, but different volumes of BSE payments will separate them forever. The characteristics of the levered firm and the unlevered one will drift from one another farther and farther with each new BSE payment.

As one can see, all conclusions of MM Propositions follow from the perfect market assumptions joined with *the implicit assumption that the value distribution is a GBM-solution with $P = 0$*, and not only from the perfect market assumptions as most economists believe now. Market friction can shift the market equilibrium to another level and change the time of market relaxation to the equilibrium. However, the friction cannot change the *sign* of the effect of debt on the firm value from positive (MMP3) to negative one (EMM; GBM, $P = \delta X$). A series of papers on the optimal capital structure based on GBM (Leland, 1994a & b; Leland & Toft, 1996; Goldstein et al., 2001; Strebulaev, 2007; Titman & Tsyplakov, 2007; Hugonnier et al., 2015, etc.) convincingly demonstrates it. These papers differ mainly in the equilibrium location spreading from 75-95 percent of debt in firm's assets (Leland, 1994a) to rather low values (Titman & Tsyplakov 2007, Hugonnier et al., 2015) depending on the type and the intensity of market friction considered in a model. All those authors believe in the positive effect of debt on the firm value; they use different types of friction to adjust the "optimal asset structure" bringing it closer to the asset structures observed in practice. The heuristic extension of GBM, $P = 0$, to JDPs by introducing random Poisson jumps in the firm value merely adds another kind of friction to GBM. This type of friction improves the normality of the GBM-distribution to a certain extent (e. g. Hilberink & Rogers 2002; Kou 2002; Chen & Kou 2009), but JDP models still bear the stigma of the GBM, $P = 0$ and the positive effect of debt on the firm value.

Discussing an influence of MM Propositions on the development of economic ideas, Pagano remarks: "Black and Scholes (1973) relied on MM-type arbitrage arguments to derive their celebrated option pricing formula and, as noted and elegantly shown by Miller (1988) himself "the familiar Put-Call Parity Theorem […] is really nothing more than the MM Proposition I only in mildly concealing disguise!"" (Pagano 2005, p. 243). The irony here is that MM Propositions are generally false, but the Black-Scholes' option pricing formula is good. The cause of this dramatic difference lies in the origin of objects they consider: Black and Scholes study option prices while Modigliani and Miller estimate the firm value. Because options are short-living objects (their expiration time is mostly 60 or 90 days), the firm value, standing behind the stocks and options, does not change perceptibly and can be considered as constant in expiration time intervals. Thus, the option price is independent of the firm value and risks. Option prices do follow GBM, $P = 0$ making the Black-Scholes formula correct. On the contrary to the options, a firm is a long-living object, and the firm's payments are essential for time intervals exceeding a year. Therefore, GBM is generally invalid for the firm, which makes the Modigliani-Miller Propositions wrong.

Conclusion

The paper considers the problem of what statistical distribution is relevant for the firm value. We show that traditional distributions, namely, the normal distribution following from the Brownian model, and the lognormal distribution following from the geometric Brownian model, GBM, lead to false conclusions on the firm value. The paper suggests an extension of the Merton model of 1974 (EMM) taking account of the firm's payments as an arbitrary function of time, and a new statistical distribution following from that model. In an open space (x, t), $x = \ln(RX/P)$, X – the firm value, t – time, R – the expected rate of returns, P – payments, the log-value distribution evolves from a normal distribution to negatively skewed one, and its skewness and variance grow with an acceleration over time. The principal difference between the lognormal GBM-distribution and the skewed EMM-distribution is the following. The GBM-firm remains "ever young", keeping invariable its mean year return R and volatility $C^{1/2}$; the random walk of the firm value is symmetric, and the default probability grows slowly. The EMM-firm, on the contrary, by and by "grows old": its effective mean year returns $R_{eff}(t)$, $0 < R_{eff}(0) < R$, is a non-increasing monotone function of time, and its effective volatility $C^{1/2}_{eff}(t) = (dVAR/dt)^{1/2}$, $C^{1/2}_{eff}(0) = C^{1/2}$, is an increasing monotone function of time, and its skewness is a negatively decreasing monotone function of time. The random walk of the firm value is now more intensive than in the GBM-case and asymmetric: a negative shift in the firm value is more likely than a positive shift; this imbalance increases the default probability. The EMM-firm has lesser longevity than the GBM-firm. EMM helps to analyze the firm's longevity and stability as a function of parameters of its business conditions and to choose the best measures feasible to the firm to strengthen its market position.

We show that MM Propositions follow, along with the perfect market assumptions, from GBM with no firm's payments, and, in general, they are false for firms paying their fixed costs, debts, taxes, and dividends (the business securing expenses, BSEs). However, for short-time deals (t << 1 year), when the mean value is about constant, MMP1 (the firm value is independent of the firm's asset structure) holds true. For longer times when the BSE payments are essential, the firm value *depends* on the firm's asset structure, and MMP1 is wrong. The other two Propositions (MMP2: the dividend policy does not affect the firm value, and PPM3: the levered firm value equals the value of the identical unlevered firm plus the present value of a tax shield) consider time intervals exceeding one year. For such intervals, BSE payments become essential, GBM is invalid, and MMP2 and MMP3 never hold good. Remarkable collateral from the inefficiency of MM Propositions is that the problem of the optimal capital structure in its traditional statement is inexistent, and the trade-off theory is wrong.

The no-arbitrage pricing principle follows from the firm's ability to keep its mean year returns as constant over time. This is always correct at the GBM-market consisting of firms making no payments or of firms whose BSE payments are proportional to the firm value. When a model takes account of BSE payments of general form (as EMM does), then for sufficiently long times the firm's *mean year returns* begin to decrease. In such conditions, there are no martingale measures and risk-neutral probabilities, and the no-arbitrage pricing principle is invalid at the market as a whole. At that, it is invalid for both modes of payments, continuous or discrete. At the market of firms paying their BSEs, a firm can keep its mean year returns about constant for the time interval $0 \leq t \leq T_{NA}$ depending on the firm's business conditions. In other words, the no-arbitrage pricing principle is valid for individual firms and only in their specific time intervals $(0, T_{NA})$, rather than for the entire market and all times as the scientific public believe now. It is interesting that in comparison with MM Proposition I, the no-arbitrage pricing principle shows more power: it is correct for times longer than a year for most firms traded at the market, while MMP1 is right only for times t << 1 year. For short-term deals with $t \ll T_{NA}$, when traders buy assets and soon resell them trying to profit on the asset price difference, the no-arbitraging principle always holds; the market is "fair" for such traders. For long-term investors, the picture is quite different. A class of long-term investors (with investment times $T_{inv} \approx T_{NA}$) includes such significant investors as pension funds, mutual funds, insurance companies, banks, and big firms. For such investors, the effects of BSE payments are essential, and the investors must watch their possible losses as the mean year returns of the firm they invested in decrease over time for $t \geq T_{NA}$. The martingale behavior of stock prices and the effectiveness of the no-arbitrage pricing principle depend on the characteristics of a firm issuing the stock. An increase in debts, interests, taxes, inflation, etc. happening within the investment period decrease the mean year returns of the firm inflicting losses to its investors. The most direct collateral from our study is that one cannot use risk-neutral probabilities when studying the firm's credit risks and default.

The model can be helpful to the firm's management for a better understanding of the current state of the firm and its prospects, especially when planning long-term business operations financed with debt capital. It can also be useful for long-term investors keeping firm stocks for a long time ($T_{inv} \approx T_{NA}$), and for banks and insurance companies estimating credit risks for a particular commercial borrower at the interval of debt maturity.

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Appendix

Here we consider the case of an unlevered firm (*UL*) and an identical levered firm (*L*) with payments $P_{UL} = \delta_1 X$, and $P_L = \delta_2 X$, ($\delta_2 > \delta_1$), and taxes of the rate *tax*. Our task is to show that in GBM, $P = \delta X$, for any reasonable firm parameters, the mean after-tax value of the unlevered firm is *greater* than the mean after-tax value of the levered firm

$$\langle X_{AT1}(t)\rangle - \langle X_{AT2}(t)\rangle = tax(\langle X_1(t-1)\rangle - \langle X_2(t-1)\rangle) + (1-tax)(\langle X_1(t)\rangle - \langle X_2(t)\rangle) - - tax(\delta_2 - \delta_1)(\langle X_2(t)\rangle - \langle X_2(t-1)\rangle) > 0 \,, \tag{8.3}$$

in complete contradiction with MMP3 (Modigliani & Miller 1963). X1(t) is the value of the unlevered firm, X2(t) is the firm value of the levered firm.

The firm development is described by Eq. (2.1), and in variable z = ln X the probability distribution for GBM, $P = \delta X$, is

$$W(z,t) = (2\pi\sigma^2)^{-1/2} exp\{-(z-H)^2/(2\sigma^2)\}\,, \tag{A.1}$$

$$H(t) = H_0 + (R-\delta)t,\ \ \sigma^2(t) = \sigma_0^2 + Ct\,,\ \ R \equiv \alpha_0 - C/2\,, \tag{A.2}$$

$$H_L(0) = H_{UL}(0) = H_0\,,\ \ \alpha_L = \alpha_{UL} = \alpha_0\,,\ \ \sigma_L^2(0) = \sigma_{UL}^2(0) = \sigma_0^2\,. \tag{A.3}$$

Eq. (A.3) reflects the conditions of the firms' identity. The mean values for the unlevered firm $\langle z_1(t)\rangle$ and for the levered firm $\langle z_2(t)\rangle$ are

$$\begin{aligned}\langle z_1(t)\rangle &\equiv H_1(t) = H_0 + (R-\delta_1)t\,,\\ \langle z_2(t)\rangle &\equiv H_2(t) = H_0 + (R-\delta_2)t\,,\end{aligned} \tag{A.4}$$

and H1(t) > H2(t). The firm mean value is $\langle X(t)\rangle = \langle e^z\rangle$, and for the distribution (A.1) – (A.2) it makes

$$\langle X(t)\rangle = \exp\{H(t) + \sigma^2(t)/2\}\,. \tag{A.5}$$

Substituting Eq. (A.2) and (A3) for H(t) and σ2(t) into (A.5), for the unlevered firm $\langle X_1(t)\rangle$ and the levered firm $\langle X_2(t)\rangle$ one has

$$\begin{aligned}&\langle X_0\rangle = \exp(H_0 + \sigma_0^2/2)\,,\\ &\langle X_1(t)\rangle = \langle X_0\rangle exp\{(R + C/2 - \delta_1)t\} = \langle X_0\rangle \exp[(\alpha_0 - \delta_1)t]\,,\\ &\langle X_2(t)\rangle = \langle X_0\rangle \exp[(\alpha_0 - \delta_2)t]\,.\end{aligned} \tag{A.6}$$

Let us denote

$$\alpha_0 k_1 \equiv \alpha_0 - \delta_1\,,\ \ \alpha_0 k_2 \equiv \alpha_0 - \delta_2\,,\ \ 0 < \alpha_0 < 1,\ \ 0 < k_2 < k_1 < 1. \tag{A.7}$$

Time t is discrete: t = 1, 2, 3, … years. With new parameters k1 and k2, the differences in parentheses in the right side of Eq. (8.3) are:

$$\langle X_1(t-1)\rangle - \langle X_2(t-1)\rangle = \langle X_0\rangle\{exp[\alpha_0 k_1(t-1)] - exp[\alpha_0 k_2(t-1)]\}, \tag{A.8}$$

$$\langle X_1(t)\rangle - \langle X_2(t)\rangle = \langle X_0\rangle\{exp[\alpha_0 k_1 t] - exp[\alpha_0 k_2 t]\}\,, \tag{A.9}$$

$$\langle X_2(t)\rangle - \langle X_2(t-1)\rangle = \langle X_0\rangle\{exp[\alpha_0 k_2 t] - exp[\alpha_0 k_2(t-1)]\}. \tag{A.10}$$

Let A1 and A2 be

$$A_1 = exp(\alpha_0 k_1) \text{ and } A_2 = exp(\alpha_0 k_2)\,, \tag{A.11}$$

then the calibrated after-tax difference for year t is

$$[\langle X_{AT1}(t)\rangle - \langle X_{AT2}(t)\rangle]/\langle X_0\rangle = tax(A_1^{t-1} - A_2^{t-1}) + (1-tax)(A_1^t - A_2^t) - \\ -\alpha_0(k_1-k_2)taxA_2^{t-1}(A_2-1)\,. \tag{A.12}$$

For the first year, t = 1, one has

$$(\langle X_{AT1}(1)\rangle - \langle X_{AT2}(1)\rangle)/\langle X_0\rangle = (1-tax)(A_1-A_2) - \alpha_0(k_1-k_2)tax(A_2-1)\,. \tag{A.13}$$

The right side of Eq. (A13) is positive when

$$(A_1-A_2)/(A_2-1) > \alpha_0(k_1-k_2)tax/(1-tax)\,. \tag{A.14}$$

Expanding exponents A1 and A2, one has

$$\frac{A_1-A_2}{A_2-1} = \alpha_0(k_1-k_2)\left[1+\frac{1}{2!}\alpha_0(k_1+k_2)+\frac{1}{3!}\alpha_0^2(k_1^2+k_1k_2+k_2^2)+\cdots\right]/ \\ \left\{\alpha_0 k_2\left[1+\frac{1}{2!}(\alpha_0k_2)+\frac{1}{3!}(\alpha_0k_2)^2+\cdots\right]\right\}, \tag{A.15}$$

and Eq. (A.14) can be presented as

$$\left[1+\frac{1}{2!}\alpha_0(k_1+k_2)+\frac{1}{3!}\alpha_0^2(k_1^2+k_1k_2+k_2^2)+\cdots\right]/ \\ \left[1+\frac{1}{2!}(\alpha_0k_2)+\frac{1}{3!}(\alpha_0k_2)^2+\cdots\right] > \alpha_0k_2tax/(1-tax) \tag{A.16}$$

Comparing correspondent terms in the numerator and the denominator of the left side, one can see that the fraction in the left side is more than unit. Hence, if the right side is less than or equal to unit, then the inequality (A.16) is true. The right side of (A.16) equals one when

$$tax = tax\_1 \equiv 1/(1+\alpha_0k_2) \tag{A.17}$$

and it is less than unit for tax < tax_1. If the rate of return is $\alpha_0 \approx 0.2$, k2 ≈ 0.7, then tax_1 = 0.877. For reasonable tax rates tax < tax_1, the right side of (A.16) is less than one. So, the inequality (A.16) is true, and that makes its equivalent inequality (8.3) at the first year to be true.

For t > 1, t = 2, 3, …, the calibrated after-tax difference (A.13) is positive if

$$tax(A_1^{t-1}-A_2^{t-1}) + (1-tax)(A_1^t-A_2^t) - \alpha_0(k_1-k_2)taxA_2^{t-1}(A_2-1) = \\ = (A_1-A_2)\left\{tax\sum_{i=0}^{t-2}A_1^{t-2-i}A_2^i + (1-tax)\sum_{i=0}^{t-1}A_1^{t-1-i}A_2^i\right\} - \\ -\alpha_0(k_1-k_2)taxA_2^{t-1}(A_2-1) > 0 \tag{A.18}$$

Or, in an equivalent form,

$$(A_1-A_2)/(A_2-1) > \alpha_0(k_1-k_2)A_2^{t-1}/\left\{\sum_{i=0}^{t-2}A_1^{t-2-i}A_2^i + \frac{1-tax}{tax}\sum_{i=0}^{t-1}A_1^{t-1-i}A_2^i\right\} \tag{A.19}$$

The right side of (A.19) is an increasing monotone function of tax; so, let us estimate the highest value of the right side at tax = 1.

$$(A_1 - A_2)/(A_2 - 1) > \alpha_0(k_1 - k_2)A_2^{t-1}/\{\sum_{i=0}^{t-2} A_1^{t-2-i} A_2^i\} \quad \text{(A.20)}$$

Using the expansion (A.16), one comes to

$$\left[1 + \frac{1}{2!}\alpha_0(k_1 + k_2) + \frac{1}{3!}\alpha_0^2(k_1^2 + k_1k_2 + k_2^2) + \cdots\right] / \left[1 + \frac{1}{2!}(\alpha_0 k_2) + \frac{1}{3!}(\alpha_0 k_2)^2 + \cdots\right] > \alpha_0 k_2 A_2^{t-1}/\{\sum_{i=0}^{t-2} A_1^{t-2-i} A_2^i\} \quad \text{(A.21)}$$

As we know, the left side is greater than one. For the second year, t = 2, the right side is

$$\alpha_0 k_2 A_2 = \alpha_0 k_2 exp(\alpha_0 k_2) .$$

The first factor $\alpha_0 k_2 \ll 1$, the second is about one; therefore, the product is less than one. For $\alpha_0 \approx 0.2$, k2 ≈ 0.7, tax = 1, the right side (A.21) is 0.161. Hence, inequality (8.3) is true for t = 2.

For t = 3, the right side of (A.21) is

$$\alpha_0 k_2 A_2^2/(A_1 + A_2) < (\alpha_0 k_2/2)exp(\alpha_0 k_2) < 1,$$

because $A_1 > A_2$. It is clear, that the greater time t, the lesser is the right side of (A.21). Therefore, the inequality (8.3) is always true for reasonable (realistic) values of problem parameters.