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PRICING IMPLICATIONS OF ASSESSING RISK OF RELATIVE WEALTH OUTCOMES

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Abstract: *When relative wealth is the variable used by agents to assess the risk of their financial future perfect coherence in the pricing of various types of financial assets ensues. The use of relative wealth instead of absolute wealth to analyze risk should not be construed as an assumption that agents do not care about absolute wealth. Relative wealth is a perfect substitute for absolute wealth in the case of certainty and on a state-of-nature by state-of-nature basis; however, the use of relative wealth to assess risk over the spectrum of states-of-nature reflects a different outlook about risk. The resulting pricing kernel and asset pricing model are reasonably robust to changes in utility functions and returns' distributions. The excess return of an asset depends on the covariance of its return with the return on the market portfolio and on all their higher co-moments as well. The simple gross risk-free return equals the harmonic mean of the probability distribution of simple gross market returns; an implication that does not bode well for financial markets in the current prolonged environment of low interest rates. It is straightforward to show that options on basic securities are priced using the same model.*

Keywords: *asset pricing, relative wealth, risk-free rate, equilibrium, options*

JEL: G12, G01, D03

Introduction

In most financial economics models agents are portrayed as mainly concerned about the risk of absolute wealth outcomes; in some models relative status plays a subordinate role. The current paper takes a different perspective and invokes agents who assess the risk of their financial future in terms of the risk of relative wealth outcomes. Using relative wealth instead of absolute wealth as the variable of interest when analyzing risk, does not mean that an assumption is being made that agents do not care about absolute wealth. They do, and relative wealth is a perfect substitute for absolute wealth in the case of certainty and on a state-of-nature by state-of-nature basis since a higher level of one is always associated with a higher level of the other because total wealth in the economy is precisely specified in such cases. However, the use of relative wealth to assess risk over the spectrum of states-of-nature reflects a different outlook about risk. The utility functions used in finance models are not direct utility functions (the numeraire-single-consumption-good assumption does not render them as such). These functions are essentially structures devised for characterizing attitudes towards risk which have there origins in attempts to solve the famous St. Petersburg's Paradox. There is no logical rationale for limiting such characterization to concerns over uncertainty of absolute wealth. Faced with uncertainty, survival and fitness are the paramount concerns. Relative wealth is a better measure for capturing these concepts than absolute wealth.

Motivation for assessing risk of relative wealth outcomes

Some results from experimental economics are supportive of the argument that relative wealth is an important consideration for agents. Rejections of offers perceived as unfair in basic 'ultimatum game' experiments[1] can be explained as attempts to avoid deterioration in relative wealth positions. People are found to care more about gains and losses than about absolute levels of wealth. If one realizes that one agent's gain is another agent's loss then this is supportive of relative wealth as an important consideration. The fact that relative pricing, not absolute pricing, is the main underpinning of markets is also supportive of the argument[2]. Neuroscience research shows that the human brain responds to perceptions of status and social hierarchies (Fliessbach et al, 2007; Zink et al, 2008; Beasley et al, 2012; Swencionis and Fiske, 2014).

Rationale for using total market value as denominator of relative wealth variable

Choosing the total market value as the basis for relative wealth assessments, as in the current paper's model, needs further motivation. All investors, regardless of size, can be characterized as averse to economic inequality. Concern for relative status is technically different from aversion to inequality (Tricomi et al, 2010); nevertheless both would have similar manifestations in the vast majority of people, who perceive a very small societal stratum that is excessively more advantaged. All societies in the world have such a demographic characterization (Piketty, 2014). A related argument is that using average wealth in the economy (total market value divided by population size) as a denominator of the relative wealth variable does not alter the model's results. Finally, as Gali (1994) suggested "…professional 'portfolio managers' …performance is evaluated in terms of the return on their portfolio relative to the rest of managers and/or the market."

De Marzo et al (2008) argue that an agent's concern for relative status is likely to increase during periods of great economic/technological upheavals. They also argue that financial bubbles can be partially explained by agents' concerns over relative wealth because they tend to trade in the same direction as the rest of the crowd out of fear of loosing their relative wealth position. Bubbles reflect a major form of market inefficiency. Relative wealth concerns, however, can lead to market inefficiency in less drastic ways. An agent might be unwilling to react completely to a new piece of information until he/she sees how other agents are going to react. This need for reinforcement might result in delays in discounting the full effect of the information in prices. Atolia and Prasad (2011) argue that relative wealth concerns "…lead to an increase in entrepreneurship and risk taking."

Several researchers incorporate agents' concern for relative status in their respective models as a complement to the fundamental consumption/wealth process

[1] In the 'ultimatum game' two players can divide a given sum of money among themselves. The division process proceeds as follows. A player offers the other a share. If the player receiving the offer accepts what is being offered, the money is split accordingly. If the offer is rejected both players get nothing (e.g. Armantier, 2006). The common explanation for rejection of offers perceived as unfair revolves around, somewhat tautologically, concerns for fairness; nevertheless Hu et al (2014) argue that status influences perceptions of fairness and responses to those perceptions and show that rejection rates increase with higher status.

[2] It is difficult to forecast changes in relative prices of goods and services which cannot be captured by a general measure of inflation. Parents who have kids to put through college might not feel much comfort in that their portfolio is expected to earn a high rate of return, which decently exceeds the expected rate of general inflation, if tuition expenses are expected to increase at a much higher rate. Pension funds face a similar situation. Assessing portfolios based on the risk of relative wealth outcomes might make these agents more assured of keeping up with the prices that are relevant.

concern (Abel, 1990; Gali, 1994; Cole et al, 1995; Bakshi and Chen, 1996; Gomez, 2007; DeMarzo et al, 2008). The role of the relative status variables is usually confined to capturing some partial concern of agents, such as ‘keeping up, or catching up, with the Joneses’ dispositions. The rest of the paper is organized as follows. Section II develops the general implications of the framework and the Relative Wealth CAPM (RWCAPM). Section III provides a straightforward illustration of the equivalence of pricing options using either the RWCAPM or the Binomial Option Pricing Model (BOPM). Section IV provides a comparison of the RWCAPM and the basic CAPM. Section V provides some final remarks.

The Model

The fundamentals of the framework

The one-period economy consists of *n* risky assets, a risk-free asset, and *m* agents. Each agent maximizes the expected utility of end-of-period relative wealth.
Agent *i*'s maximization program is as follows:

$$Max\,E\left[U^i\left(\frac{\left\{\sum_{j=1}^{n}\alpha_j^i\,\tilde{P}_{1j}+K^i\left(1+r_f\right)\right\}}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right)\right]$$

$$subject\ to\ \ W_0^i=K^i+\sum_{j=1}^{n}\alpha_j^i P_{0j}$$

The Lagrangian for this program is as follows:

$$Max\,E\left[U^i\left(\frac{\left\{\sum_{j=1}^{n}\alpha_j^i\,\tilde{P}_{1j}+K^i\left(1+r_f\right)\right\}}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right)\right]+\gamma\left[W_0^i-K^i-\sum_{j=1}^{n}\alpha_j^i P_{0j}\right]$$

P_{1j} represents the end-of-period random payoffs of asset *j*. The ~ represents randomness.
K^i is the amount that agent *i* decides to invest in the risk-free asset whose rate of return is r_f .
W^i_0 is the beginning-of-period wealth for agent *i*.
P_{0j} is the beginning-of-period price of asset *j*.
α^i_j is the fraction of asset *j* that agent *i* decides to hold. Short selling is allowed.
γ is the Lagrangian multiplier.

Note that the end-of-period random relative wealth[3] for agent *i*, RW^i_1 , equals:

$$RW_1^i = \frac{\left\{\sum_{j=1}^{n} \alpha_j^i \tilde{P}_{1j} + K^i\left(1+r_f\right)\right\}}{\sum_{j=1}^{n} \tilde{P}_{1j}}$$

The first order conditions with respect to K^i , and α^i_j ,respectively, are as follows:

$$E\left[\frac{\partial U^i}{\partial RW_1^i}\frac{\left(1+r_f\right)}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]-\gamma=0 \qquad (1)$$

$$E\left[\frac{\partial U^i}{\partial RW_1^i}\frac{\tilde{P}_{1j}}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]-\gamma P_{0j}=0 \qquad (2)$$

From equations (1) and (2):

$$E\left[\frac{\partial U^i}{\partial RW_1^i}\frac{\left(\tilde{P}_{1j}-P_{0j}\left(1+r_f\right)\right)}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]=0 \quad \forall j \qquad (3)$$

Multiplying both sides by $\dfrac{\sum_{j=1}^{n}P_{0j}}{P_{0j}}$

$$E\left[\frac{\partial U^i}{\partial RW_1^i}\frac{\left(\tilde{R}_j-r_f\right)}{\left(1+\tilde{R}_m\right)}\right]=0 \quad \forall j \qquad (4)$$

R_j and R_m are the random returns of asset *j* and the market portfolio respectively, where:

$$\tilde{R}_j=\frac{\tilde{P}_{1j}-P_{0j}}{P_{0j}} \quad and \quad \tilde{R}_m=\frac{\sum_{j=1}^{n}\tilde{P}_{1j}-\sum_{j=1}^{n}\tilde{P}_{0j}}{\sum_{j=1}^{n}\tilde{P}_{0j}}$$

Using definition of covariance:

$$E\left[\frac{\partial U^i}{\partial RW_1^i}\right]E\left[\frac{\tilde{R}_j-r_f}{1+\tilde{R}_m}\right]=-Cov\left(\frac{\partial U^i}{\partial RW_1^i},\ \frac{\tilde{R}_j-r_f}{1+\tilde{R}_m}\right) \qquad (5)$$

Let's begin the analysis assuming each agent has quadratic utility as follows:

$$U^i = a^i RW_1^i - \frac{A^i}{2}\left(RW_1^i\right)^2$$

[3] To simplify notation the ~ symbol, which represents randomness, is not placed over the RW^i_1 variable.

Then:

$$\frac{\partial U^i}{\partial RW_1^i} = a^i - A^i RW_1^i, \qquad \text{and} \qquad \frac{\partial^2 U^i}{\partial {RW_1^i}^2} = -A^i$$

Therefore, from equation (5):

$$E\left[a^i - A^i RW_1^i\right] E\left[\frac{\tilde{R}_j - r_f}{1+\tilde{R}_m}\right] = A^i Cov\left(RW_1^i, \frac{\tilde{R}_j - r_f}{1+\tilde{R}_m}\right) \qquad (6)$$

Agent's i absolute risk aversion parameter, φ^i, is given by:

$$\varphi^i = -E\left[\frac{\partial^2 U^i}{\partial {RW_1^i}^2}\right] \Big/ E\left[\frac{\partial U^i}{\partial RW_1^i}\right] = A^i \Big/ E\left[a^i - A^i RW_1^i\right]$$

Therefore, from equation (6):

$$\left(\varphi^i\right)^{-1} E\left[\frac{\tilde{R}_j - r_f}{1+\tilde{R}_m}\right] = Cov\left(RW_1^i, \frac{\tilde{R}_j - r_f}{1+\tilde{R}_m}\right) \qquad (7)$$

Summing equation (7) across all m agents in the economy and noting that since[4] $\sum_{i=1}^m RW_1^i = 1$ then the aggregated covariance term equals zero and noting that $\sum_{i=1}^m \left(\varphi^i\right)^{-1} \neq zero$:

$$E\left[\frac{\tilde{R}_j - r_f}{1+\tilde{R}_m}\right] = 0 \qquad (8)$$

Another form of equation (8) is:

$$E\left[\frac{\left(\tilde{P}_{1j} - P_{0j}\left(1+r_f\right)\right)}{\sum_{j=1}^n \tilde{P}_{1j}}\right] = 0 \qquad (9)$$

because multiplying both sides by $\frac{\sum_{j=1}^n P_{0j}}{P_{0j}}$ leads to the original form.

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[4] $\sum_{i=1}^m W_t^i = \sum_{j=1}^n P_{tj} \; for \; t = 0,1 \; because \sum_{i=1}^m \alpha_j^i = 1 \forall j \; and \; \sum_{i=1}^m K^i = 0$

From definition of covariance:

$$E\left[\tilde{R}_j\right] - r_f = \frac{-Cov\left(\tilde{R}_j, \frac{1}{1+\tilde{R}_m}\right)}{E\left[\frac{1}{1+\tilde{R}_m}\right]} \tag{10}$$

The intuition for equation (10) is as follows: $1+\tilde{R}_j$ is the random payoff at time one from an investment of one dollar in asset *j* at time zero. $\frac{1}{1+\tilde{R}_m}$ is the price that would be recognized, at time one, to have been paid at time zero corresponding to a one dollar payoff from the market portfolio when a given value of $\tilde{R}_m$ materializes. As the covariance between $1+\tilde{R}_j$ and $\frac{1}{1+\tilde{R}_m}$ increases (becomes less negative) asset *j* will be more valuable because the payoffs from investing in *j* would be a better counter to the prices that would have been paid for a given payoff from the market portfolio. $E\left[\frac{1}{1+\tilde{R}_m}\right]$, in the denominator of the R.H.S., reconciles the time frames of the two elements of the covariance because, as shown in the next subsection, its inverse is equal to $1+r_f$.

Multiplying both sides of equation (10) by the weight of asset *j* in the market portfolio and summing over all *n* assets we get:

$$E\left[\tilde{R}_m\right] - r_f = \frac{-Cov\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right)}{E\left[\frac{1}{1+\tilde{R}_m}\right]} \tag{11}$$

$$E\left[\frac{1}{1+\tilde{R}_m}\right] = \frac{-Cov\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right)}{E\left[\tilde{R}_m\right] - r_f} \tag{12}$$

Substituting in equation (10) we get:

$$E\left[\tilde{R}_j\right] - r_f = \left\{ \frac{Cov\left(\tilde{R}_j, \frac{1}{1+\tilde{R}_m}\right)}{Cov\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right)} \right\} \left(E\left[\tilde{R}_m\right] - r_f\right) \tag{13}$$

Equation (13) is the Relative Wealth Capital Asset Pricing Model (RWCAPM).

In brief form the RWCAPM is expressed as follows:

$$E\left[\tilde{R}_j\right] - r_f = \chi_{jm}\left(E\left[\tilde{R}_m\right] - r_f\right) \tag{14}$$

$$\chi_{jm} = \left\{ Cov\left(\tilde{R}_j, \frac{1}{1+\tilde{R}_m}\right) \Big/ Cov\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right) \right\} \qquad (15)$$

It is clear that $\chi_{mm} = 1$.

The model is reasonably robust to deviations from assumption of quadratic utility

For other utility functions (e.g. *Logarithmic* $U^i = a^i \ln\left(RW_1^i\right)$, $\frac{\partial U^i}{\partial RW_1^i} = \frac{a^i}{RW_1^i}$), if all agents have the same utility function (e.g. they all have a logarithmic function with the same a^i) then the RWCAPM holds approximately. The reason is that $\sum_{i=1}^{m} RW_1^i = 1$, i.e. it is fixed. It can be shown with numerical examples that for the condition:

$$E\left[f\left(\tilde{X}^i\right)\right]E\left[\tilde{Y}\right] = -Cov\left[f\left(\tilde{X}^i\right), \tilde{Y}\right]$$

to apply to each member *i* of the population (which is the scheme of equation (5)) when $\sum_{i=1}^{m} \tilde{X}^i$ is constant then $E\left[\tilde{Y}\right]$ is usually very small compared to the absolute value of any single value that *Y* can assume. That is, $E\left[\tilde{Y}\right]$ is *effectively* zero. Numerical examples do not, of course, represent proof but provide tentative support to the argument that the model is robust to changes in the utility functions.

Carrying out the analysis from a representative agent's perspective leads to the RWCAPM regardless of the utility function. Looking at equation (5) from a representative agent's perspective it is noted that RW_1 is fixed and hence also is $\frac{\partial U}{\partial RW_1}$ (regardless of the utility function) thus the covariance term on the R.H.S. vanishes. The result is equation (8) leading to the RWCAPM.

It is interesting that equation (9) also obtains in the case of a representative agent with logarithmic preferences over absolute wealth. For this case the following equation replaces equation (5)

$$E\left[\tilde{R}_j - r_f\right] = -Cov\left(\frac{\partial U}{\partial W_1}, \tilde{R}_j\right) \Big/ E\left[\frac{\partial U}{\partial W_1}\right] \qquad where\ \frac{\partial U}{\partial W_1} = \frac{a}{W_1}$$

Multiplying both the numerator and denominator by the value of aggregate wealth at time zero leads to equation (9) and hence to the RWCAPM. It is difficult though to ascertain the significance of this fact. Maximizing the expected logarithm of wealth is argued for by many researchers (e.g. Kelley, 1956; Bell and Cover, 1980; Evstigneev et al, 2008) as the best strategy for survival and dominance[5]. However, this strategy is advocated from the perspective of individual agents not from that of a representative agent.

Further details

Some implications of the model

It is evident from equation (10) that $1/\left(1+\tilde{R}_m\right)$ represents the pricing kernel (or stochastic discount factor) M_1 underlying the model. Equation (10) is a special case of the well known relationship for a pricing kernel:

$$E\left[\tilde{R}_j - r_f\right] = -\frac{Cov\left(M_1, \left(\tilde{R}_j - r_f\right)\right)}{E\left[M_1\right]} \tag{16}$$

From equation (8)

$$r_f = \frac{E\left[\frac{\tilde{R}_j}{1+\tilde{R}_m}\right]}{E\left[\frac{1}{1+\tilde{R}_m}\right]} \tag{17}$$

Similarly, multiplying equation (8) by the weight of asset j in the market portfolio and summing over all n assets and rearranging we get:

$$r_f = \frac{E\left[\frac{\tilde{R}_m}{1+\tilde{R}_m}\right]}{E\left[\frac{1}{1+\tilde{R}_m}\right]} = \left\{\frac{1}{E\left[\frac{1}{1+\tilde{R}_m}\right]}\right\} - 1 \tag{18}$$

Equation (18) provides a relationship between the risk-free rate and moments of the probability distribution of market returns. Equation (18) can be rewritten as:

$$1 + r_f = \frac{1}{E\left[\frac{1}{1+\tilde{R}_m}\right]} \qquad where \quad 0 < E\left[\frac{1}{1+\tilde{R}_m}\right] \le 1 \tag{19}$$

Thus the simple gross risk-free return is the (weighted) harmonic mean[6] of the probability distribution of simple gross market returns.

The implication embodied in equation (19) does not bode well for financial markets performance if the risk-free rate continues to be depressed by the central banks of most

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[5] Sinn and Weichenrieder (1993) write "…nature links the generational risks not according to an additive, but according to a multiplicative function," and a multiplicative function can be transformed into an additive one by taking logarithms.

[6] The weights of the harmonic mean are the relevant probability measures. $1/\left(1+\tilde{R}_m\right)$ can be interpreted as the present value of one dollar to be received at the end of the period using R_m as the discount factor. $E\left[1/\left(1+\tilde{R}_m\right)\right]$ is the mean of these present values.

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developed economies, as it has been since the turmoil of 2008. Either the market return stabilizes in a narrow band around a very low mean which is highly unlikely or becomes excessively volatile which is the more probable scenario in the long term as central bank interventions become less and less effective. This theoretical link between very low interest rates and inevitable excessive volatility resonates with market microstructure and behavioral factors discussed by El-Erian (2016) that "…have been turbocharged by the low interest rate environment" and that make markets inherently prone to excessive volatility.

As a simple back-of-the-envelop illustration, assume that the market return can take any of the following four values: -25%, -20%, 20%, 50% with equal probabilities. From these values, the expected market return is 6.25%; using equation (19), the risk-free rate is -2%. El-Erian (2016) notes that "…a sizeable amount of government bonds in Europe is trading at negative nominal yields," which means that investors are "…agreeing to pay (rather than receive) interest income."

With the help of equation (19), equation (9) can be recast as

$$E\left[\frac{\tilde{P}_{1j}}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]-\frac{P_{0j}}{\sum_{j=1}^{n}P_{0j}}=0 \quad \textit{alternatively} \quad \frac{E\left[\frac{\tilde{P}_{1j}}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]}{\frac{P_{0j}}{\sum_{j=1}^{n}P_{0j}}}-1=0$$

This implies according to Long (1990) that the market portfolio is a numeraire portfolio[7] and that there are 'no profit opportunities' i.e. the no-arbitrage condition prevails, implying equilibrium. In equilibrium the expected relative contribution of asset *j* to aggregate wealth at time one is equal to its certain relative contribution at time zero. This is the result of interactions between agents all of whom assess utility over relative wealth.

A general restriction of the model is:

$$E\left[\frac{r_f}{1+\tilde{R}_m}\right]=E\left[\frac{\tilde{R}_m}{1+\tilde{R}_m}\right]=E\left[\frac{\tilde{R}_j}{1+\tilde{R}_m}\right]=\frac{r_f}{1+r_f} \quad \forall j \qquad (20)$$

The last equality in equation (21) results from equation (20). Thus

$$E\left[\frac{1+r_f}{1+\tilde{R}_m}\right]=E\left[\frac{1+\tilde{R}_m}{1+\tilde{R}_m}\right]=E\left[\frac{1+\tilde{R}_j}{1+\tilde{R}_m}\right]=1 \qquad (21)$$

The expected value of the ratio of payoffs from a one dollar investment in any asset and a one dollar investment in the market portfolio is the same for all assets and equals one. This is the result of relative wealth being the emphasis of agents and their need for reassurance that their individually chosen portfolio composition will not adversely affect their relative wealth position. Equations (20) and (21) do not restrict the returns of different assets to be close to each other; rather, they reflect restrictions on the structures of returns as a whole. As an

[7] Long (1990) states "An asset list offers no profit opportunities if and only if a numeraire portfolio can be formed from the list. A numeraire portfolio is defined to be a self-financing portfolio such that, if current and future asset prices and dividends are denominated in units of the numeraire (that is, divided by the contemporaneous value of the numeraire portfolio), the expected rate of return of every asset on the list is always equal to zero."

illustration, for the numerical example used earlier in this sub-subsection where the market returns can take any of the following four values: -25%, -20%, 20%, 50% with equal probabilities, this can correspond to two market segments, with equal weights, where one segment's returns can take the values -35%, -30%, 35%, 70%

and the other segment's returns can take the values -15%, -10%, 5%, 30%. These values satisfy the restrictions in equations (20) and (21).

Equations (19) and (21) show that $\frac{1}{1+\tilde{R}_m}$, as a pricing kernel M_1, satisfies the following additional well known relationships:

$$E[M_1] = \frac{1}{(1+r_f)} \qquad (22)$$

$$E[M_1(1+\tilde{R}_j)] = 1 \qquad (23)$$

The model encompasses preferences for higher moments

A Taylor expansion of $\frac{1}{1+\tilde{R}_m}$ around zero yields: $\frac{1}{1+\tilde{R}_m} = 1 - \tilde{R}_m + \tilde{R}_m^2 - \tilde{R}_m^3 + \tilde{R}_m^4 - \tilde{R}_m^5 + \ldots$

The pricing kernel underlying the RWCAPM is "highly nonlinear" and decreasing in R_m . These characteristics of a pricing kernel are advocated by several researchers (e.g. Dittmar, 2002). Using the Taylor expansion in equation (13) indicates that the excess return of asset j does not depend only on the covariance of its return with R_m , as in the basic CAPM, but also on the coskewness, cokurtosis, and all the higher co-moments. All of these co-moments[8], are encompassed by covariance with $\frac{1}{1+\tilde{R}_m}$. Noting that the covariance in the denominator of χ_{jm} is always negative, it can be seen that a market populated by agents concerned about the risk of relative wealth outcomes shows aggregate preference for low covariance, co-kurtosis and all other even co-moments, between R_j and R_m, while it prefers high co-skewness and all other odd co-moments - in line with Scott and Horvath (1980).

Whereas the basic CAPM emerges from the development of the 'expected returns-variance of returns (E-V) rule' established by Markowitz (1952), the RWCAPM can be seen as emerging from an 'expected returns-moments of returns (E-Mom) rule'. The higher moments, beyond variance, are important because investors are concerned about the risk of relative wealth outcomes. In developing the basic CAPM, Sharpe (1964) assumes agents have a utility function of the form $U = f(E_W, \sigma_W)$ where E_W is the expectation of future wealth and σ_W its standard deviation. The RWCAPM can be seen as based on agents having a utility function of the form $U = f(E_W, -\text{cov}(\tilde{W}, \frac{1}{\tilde{W}})$ where the covariance term is the risk measure that encompasses all of the moments (second degree and higher) of the future-wealth distribution. Appendix C utilizes this form of the utility function in an analysis that provides further support to the argument that the RWCAPM is an equilibrium model.

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[8] The importance of higher moments, especially skewness and kurtosis, has been highlighted by many researchers, e.g. Kraus and Litzenberger (1976) and Jondeau and Rockinger (2006).

Chi for an asset, χ_{jm} , can be greater than, equal, or less than its beta β_{jm} depending on the interplay between the co-moments beyond covariance in both the numerator and denominator of χ_{jm}. The relationship between the two parameters, which results from simple statistical manipulations, is as follows:

$$\chi_{jm} = \frac{\dfrac{E\left[\tilde{R}_m\right]E\left[\dfrac{\tilde{R}_j}{1+\tilde{R}_m}\right]}{E\left[\dfrac{1}{1+\tilde{R}_m}\right]Var\left(\tilde{R}_m\right)} - \dfrac{E\left[\tilde{R}_j\,\tilde{R}_m\right]}{Var\left(\tilde{R}_m\right)} + \beta_{jm}}{\dfrac{E\left[\tilde{R}_m\right]E\left[\dfrac{\tilde{R}_m}{1+\tilde{R}_m}\right]}{E\left[\dfrac{1}{1+\tilde{R}_m}\right]Var\left(\tilde{R}_m\right)} - \dfrac{E\left[\tilde{R}_m^2\right]}{Var\left(\tilde{R}_m\right)} + 1} \tag{24}$$

Thus χ_{jm} equals β_{jm} if:

$$E\left[\tilde{R}_j\,\tilde{R}_m\right] = \frac{E\left[\tilde{R}_m\right]E\left[\dfrac{\tilde{R}_j}{1+\tilde{R}_m}\right]}{E\left[\dfrac{1}{1+\tilde{R}_m}\right]} \quad and \quad E\left[\tilde{R}_m^2\right] = \frac{E\left[\tilde{R}_m\right]E\left[\dfrac{\tilde{R}_m}{1+\tilde{R}_m}\right]}{E\left[\dfrac{1}{1+\tilde{R}_m}\right]}$$

Or, equivalently, if:

$$Cov\left(\tilde{R}_j\,\tilde{R}_m\,,\frac{1}{1+\tilde{R}_m}\right) = Cov\left(\tilde{R}_m\,,\frac{\tilde{R}_j}{1+\tilde{R}_m}\right) \quad and \quad Cov\left(\tilde{R}_m^2\,,\frac{1}{1+\tilde{R}_m}\right) = Cov\left(\tilde{R}_m\,,\frac{\tilde{R}_m}{1+\tilde{R}_m}\right)$$

If these conditions are satisfied then the effects of the co-moments between R_j and R_m , beyond covariance, cancel each other out in both the numerator and denominator of χ_{jm} . The economic meaning of this is that any beneficial effects of positive odd co-moments (co-skewness, etc…) or negative even co-moments (co-kurtosis, etc…) are cancelled out by the detrimental effects of negative odd co-moments or positive even co-moments; in this case the basic CAPM and RWCAPM are equivalent. In essence one can argue that the basic CAPM is a special case of the RWCAPM.

Whether or not χ_{jm} equals β_{jm} depends on the return distribution of each individual asset and its interplay with the return distribution of the market portfolio. In this paper's framework there is no dominant parameter whose adjustment leads to complete congruence with the basic CAPM for *all* assets as is the case, for example, in Gali's (1994) 'basic equivalence result' which states that "…equilibrium asset prices and returns in an economy with externalities are identical to those of an externality-free economy with a properly adjusted degree of risk aversion." In fact, as discussed below, risk aversion parameters play no observable role in aggregate in the relative wealth framework.

A note on aggregate risk aversion

It is interesting to ponder the absence of an aggregate risk aversion factor in equation (10). Such a factor is very much present in the corresponding equation that materializes when the outcome variable of interest, for risk assessment, is wealth rather than relative wealth. The analysis in Appendix B shows that the aggregate ARA factor acts in two offsetting ways. First, it is a pricing factor for the covariance between an asset's returns and the reciprocals of the simple gross market returns. Second, it is a reducing factor for the magnitude of this same covariance. These offsetting actions lead to a constant effective aggregate ARA (equal to 1). This fits well with the fact that the aggregate relative wealth invested in risky assets is constant at a value of one (thus aggregate relative risk aversion equals aggregate ARA). The analysis in Appendix B also shows that the absence of the risk aversion factor occurs not only at the aggregated level but also for the individual agent when there is a single risky asset and a single risk-less asset.

Thus in the relative wealth framework risk aversion is mainly a motivator at the individual agent level for calibration of risk premiums *across* assets until each asset satisfies equation (10). The importance of this calibration is intuitively the result of relative wealth being the emphasis of each agent and his need for reassurance that his individually chosen portfolio composition will not adversely affect his relative wealth position. In the relative wealth framework the aggregate risk aversion factor is not assigned the job of determining the market risk premium since the risk free rate is determinable endogenously as is shown in a previous sub-subsection.

A note on empirical estimation

Empirical estimates for χ_{jm} for an asset j can be obtained by regressing

$\tilde{R}_j$ *on* $\frac{1}{1+\tilde{R}_m}$ and regressing $\tilde{R}_m$ *on* $\frac{1}{1+\tilde{R}_m}$ and dividing the

slope coefficient of the first regression by the slope coefficient of the second regression. Several researchers note that during periods of financial instability assets' return distributions exhibit large deviations from normality and/or symmetry (Pownall and Koedijk, 1999; Lillo and Mantegna, 2000; Consigli, 2002). As shown with numerical examples in Appendix A, for skewed distributions the χ_{jm} factor is significantly higher than the β_{jm} factor for negatively skewed distributions and significantly lower for positively skewed distributions.

Equivalence of RWCAPM and BOPM

This section provides a straightforward illustration of the fact that asset pricing models satisfying the no-arbitrage condition are equivalent to option-pricing models.
Such illustrations are much more elaborate with pricing models other than the RWCAPM; for example, Hsia (1981) invokes several assumptions to establish this equivalence.

In a binomial state-of-nature framework if the binomial trees for basic securities (including the risk-free bond) satisfy the general restriction presented in equations (20) and (21) then the pricing of options on those basic securities using the RWCAPM is equivalent to their pricing using the Binomial Option Pricing Model (BOPM).

Let the following represent the binomial tree for basic security *j* :

$$P_{1jU} \quad \textit{the higher price at end of period} \quad, \textit{probability of this state is } q$$

$$P_{0j}$$

$$P_{1jD} \quad \textit{the lower price at end of period} \quad, \textit{probability of this state is } (1-q)$$

The price of a call option *C* on security *j* is determined using the BOPM as follows:

$$C_0 = \frac{\left(\frac{(1+r_f)-\frac{P_{1jD}}{P_{0j}}}{\frac{P_{1jU}}{P_{0j}}-\frac{P_{1jD}}{P_{0j}}}\right)C_{1U} + \left(\frac{\frac{P_{1jU}}{P_{0j}}-(1+r_f)}{\frac{P_{1jU}}{P_{0j}}-\frac{P_{1jD}}{P_{0j}}}\right)C_{1D}}{(1+r_f)}$$

C_{1U} is the higher option payoff (associated with the upper branch of the tree)
C_{1D} is the lower option payoff (associated with the lower branch of the tree).

Similarly, the price of a put option *PUT* on security *j* is determined using the BOPM as follows:

$$PUT_0 = \frac{\left(\frac{(1+r_f)-\frac{P_{1jD}}{P_{0j}}}{\frac{P_{1jU}}{P_{0j}}-\frac{P_{1jD}}{P_{0j}}}\right)PUT_{1D} + \left(\frac{\frac{P_{1jU}}{P_{0j}}-(1+r_f)}{\frac{P_{1jU}}{P_{0j}}-\frac{P_{1jD}}{P_{0j}}}\right)PUT_{1U}}{(1+r_f)}$$

PUT_{1D} is the lower option payoff (associated with the upper branch of the tree)
PUT_{1U} is the higher option payoff (associated with the lower branch of the tree).
Algebraic manipulation leads to:

$$C_0 = \left(\frac{P_{0j}-\frac{P_{1jD}}{(1+r_f)}}{P_{1jU}-P_{1jD}}\right)C_{1U} + \left(\frac{\frac{P_{1jU}}{(1+r_f)}-P_{0j}}{P_{1jU}-P_{1jD}}\right)C_{1D}$$

$$PUT_0 = \left(\frac{P_{0j}-\frac{P_{1jD}}{(1+r_f)}}{P_{1jU}-P_{1jD}}\right)PUT_{1D} + \left(\frac{\frac{P_{1jU}}{(1+r_f)}-P_{0j}}{P_{1jU}-P_{1jD}}\right)PUT_{1U}$$

On the other hand, the prices of the call and put options using the RWCAPM are determined as follows:

$$C_0^* = q\frac{C_{1U}}{1+R_{mU}} + (1-q)\frac{C_{1D}}{1+R_{mD}}$$

$$PUT_0^* = q\frac{PUT_{1D}}{1+R_{mU}} + (1-q)\frac{PUT_{1U}}{1+R_{mD}}$$

R_{mU} and R_{mD} are the returns of the market portfolio in the state with probability *q* (upper branch of the tree) and the state with probability (1-*q*) (lower branch) respectively.

$$C_0 = C_0^* \; and \; PUT_0 = PUT_0^* \quad iff \quad \frac{q}{1+R_{mU}} = \frac{P_{0j} - \frac{P_{1jD}}{(1+r_f)}}{P_{1jU} - P_{1jD}} \quad and \quad \frac{1-q}{1+R_{mD}} = \frac{\frac{P_{1jU}}{(1+r_f)} - P_{0j}}{P_{1jU} - P_{1jD}}$$

Since basic securities satisfy the general restriction in equations (21) and (22)

$$P_{0j} = q\frac{P_{1jU}}{1+R_{mU}} + (1-q)\frac{P_{1jD}}{1+R_{mD}}$$

$$\frac{1}{1+r_f} = q\frac{1}{1+R_{mU}} + (1-q)\frac{1}{1+R_{mD}}$$

Plugging the above two expressions in $\frac{P_{0j} - \frac{P_{1jD}}{(1+r_f)}}{P_{1jU} - P_{1jD}}$ *and* $\frac{\frac{P_{1jU}}{(1+r_f)} - P_{0j}}{P_{1jU} - P_{1jD}}$ leads to the proof that pricing using the BOPM is equivalent to that using the RWCAPM. The put-call parity can be easily shown to prevail in the RWCAPM framework. The put-call parity entails that:

$$C_0 - PUT_0 = P_{0j} - PV(K)$$

PV(K) is the present value (discounting at the risk-free rate) of the same exercise price *K* for both the call and put options.

In the RWCAPM framework:

$$C_0 - PUT_0 = \frac{q}{1+R_{mU}}\{Max(P_{1jU} - K, 0) - Max(K - P_{1jU}, 0)\} + \frac{1-q}{1+R_{mD}}\{Max(P_{1jD} - K, 0) - Max(K - P_{1jD}, 0)\}$$

$$C_0 - PUT_0 = \frac{q}{1+R_{mU}}(P_{1jU} - K) + \frac{1-q}{1+R_{mD}}(P_{1jD} - K)$$

Also, in the RWCAPM framework:

$$P_{0j} - PV(K) = P_{0j} - \frac{K}{1+r_f} = \frac{q}{1+R_{mU}}(P_{1jU} - K) + \frac{1-q}{1+R_{mD}}(P_{1jD} - K)$$

Comparison between the RWCAPM and the basic CAPM

This section provides a comparison of the RWCAPM and the basic CAPM.

In both the RWCAPM and the basic CAPM the market portfolio is the relevant portfolio that establishes the return-risk tradeoff in the market. Note that, both χ_{jm}, of the RWCAPM, and β_{jm}, of the basic CAPM, are risk measures that relate the excess return of an asset to the excess return on the market and reflect the co-movement of the returns of asset *j* with the returns of the market portfolio. However, the measure of risk is different in the two models; in the RWCAPM the risk measure encompasses considerations for all higher co-moments between R_j and R_m (covariance and beyond) whereas in the basic CAPM the risk measure only reflects the covariance[9]. As argued before, in essence one can argue that the basic CAPM is a special case of the RWCAPM.

Both the RWCAPM and the basic CAPM can be developed intuitively, in an analogous way, from the market return-risk trade-off ratio for each.

Basic CAPM:

Return-risk trade-off:

$$\frac{E[\tilde{R}_m]-r_f}{Var(\tilde{R}_m)}=\frac{\sum_{j=1}^{n}x_j\left(E[\tilde{R}_j]-r_f\right)}{\sum_{j=1}^{n}x_j Cov(\tilde{R}_j,\tilde{R}_m)}=\frac{E[\tilde{R}_j]-r_f}{Cov(\tilde{R}_j,\tilde{R}_m)} \qquad (25)$$

The first equality is a statistical identity; the second equality is a plausible intuition since the contribution of asset *j* to both the excess return and risk of the market portfolio are weighted by its weight in the market portfolio (x_j). Equating the first and last ratios leads to the CAPM.

RWCAPM: Return-risk trade-off[10]:

$$\frac{E[\tilde{R}_m]-r_f}{-Cov\left(\tilde{R}_m\,,\frac{1}{1+\tilde{R}_m}\right)}=\frac{\sum_{j=1}^{n}x_j\left(E[\tilde{R}_j]-r_f\right)}{-\sum_{j=1}^{n}x_j Cov\left(\tilde{R}_j,\frac{1}{1+\tilde{R}_m}\right)}=\frac{E[\tilde{R}_j]-r_f}{-Cov\left(\tilde{R}_j,\frac{1}{1+\tilde{R}_m}\right)} \qquad (26)$$

The first equality is a statistical identity; the second equality is a plausible intuition since the contribution of asset *j* to both the excess return and risk of the market portfolio are weighted by its weight in the market portfolio (x_j). Equating the first and last ratios leads to the RWCAPM.

The basic CAPM has been put to many uses, such as estimating the cost of equity capital, and serving as a benchmark for fund performance. It is clear that the same purposes can be served more accurately by the RWCAPM which takes into account co-moments beyond the covariance.

The RWCAPM is reasonably robust to changes in utility functions and return distributions whereas the basic CAPM is based on restrictive assumptions (quadratic utility and/or normal distributions).

The basic CAPM does not provide any theoretical foundation for the expected return on the market portfolio; it only uses it as an anchor for the expected return on individual assets. Merton (1980) writes "…one might say that to attempt to estimate the expected return on the market is to embark on a fool's errand." He presents several intuitive, but not theoretically grounded models for this return. In contrast the RWCAPM establishes the market portfolio as a numeraire portfolio and relates the expected return on the market portfolio to the risk-free rate. This is a relationship that needs to be heeded by central banks in the current environment of very low interest rates. It is also a relationship that can provide some explanation for the acceptance of negative nominal yields currently prevailing in several countries.

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[9] Refer to the discussion in the sub-section titled 'The model encompasses preferences for higher moments' and appendix A.

[10] See the earlier discussion of the risk measure under the RWCAPM. The covariance is clearly negative, so the negative sign leads to a positive risk measure analogous to σ^2.

The basic CAPM implies that all investors hold an identical portfolio, the market portfolio. On the other hand, in the RWCAPM framework investors hold different portfolios, which is what one observes in markets. Holding zero-sum-game securities, such as derivatives, can be rationalized in the RWCAPM framework; on the other hand, it is well known that holding derivatives cannot be rationalized in the basic CAPM framework wherein all investors hold the market portfolio.

Concluding remarks

The present paper explores the implications for asset pricing if economic agents structure their risk assessments around relative wealth. Relative wealth encompasses the natural and intuitive concern for absolute wealth on a state-of-nature by state of nature basis and, concomitantly, in the case of certainty because relative wealth is a perfect substitute for absolute wealth in such cases. However, the use of relative wealth to assess risk over the spectrum of states-of-nature reflects a different outlook about risk. Reducing the risk of absolute wealth is not the best antidote to reduce the risk of survival and fitness, which is of paramount importance when faced with uncertainty. Reducing the risk of relative wealth is. This emphasis on assessing the risk of relative wealth results in coherence in the pricing of various categories of assets. An asset's risk premium is driven by the covariance of the asset's return with the reciprocal of the simple gross market return; this is equivalent to saying that the asset's risk premium does not depend only on the covariance of its return with market return, as in the basic CAPM, but also on their coskewness, cokurtosis, and all higher co-moments. The simple gross risk-free return is the harmonic mean of the probability distribution of simple gross market returns. It is straightforward to show that options on basic securities can be priced using the same model.

Future work might look, in more detail, into the implications of the model for portfolio compositions and into the implications of heterogeneous preferences. The relationship between the risk-free rate and the distribution of market returns is also an important area of research especially given the current environment of (very) low interest rates that is expected to continue for the foreseeable future.

References

Abel, A.B. (1990). Asset Prices under Habit Formation and Catching up with the Joneses. *American Economic Association Papers and Proceedings*, 80(2), 38-42.

Armantier, O. (2006). Do Wealth Differences Affect Fairness Considerations? *International Economic Review*, 47(2), 391-429.

Atolia, M. and Prasad, K. (2011). Relative Wealth Concerns and Entrepreneurship. *Economica*, 78, 294-316.

Bakshi, G. and Chen, Z. (1996). The Spirit of Capitalism and Stock-Market Prices. *The American Economic Review*, 86(1), 133-157.

Beasley, M., Sabatinelli, D., Obasi, E. (2012). Neuroimaging Evidence for Social Rank Theory. *Frontiers in Human Neuroscience*, 6, (May).

Bell, R.M., and Cover, T. M. (1980). Competitive Optimality of Logarithmic Investment. *Mathematics of Operations Research*, 5(2).

Cole, H.L., Mailath, G.J., and Postlewaite, A. (1995). Incorporating Concern for Relative Wealth into Economic Models. *Federal Reserve Bank of Minneapolis Quarterly Review*, 19(3).

Consigli, G. (2002). Tail Estimation and Mean-VAR Portfolio Selection in Markets Subject to Financial Instability. *Journal of Banking & Finance*, 26, 1355-1382.

Copeland, T.E. and Weaton, J.F. (1992). *Financial Theory and Corporate Policy* (3rd ed.). (Addison-Wesley, Reading, Massachusetts)

DeMarzo, P.M., Kaniel, R., and Kremer, H. (2008). Relative Wealth Concerns and Financial Bubbles. *Review of Financial Studies*, 21(1), 19-50.

Dittmar, R.F. (2002). Nonlinear Pricing Kernels, Kurtosis Preference, and Evidence from the Cross Section of Equity Returns. *Journal of Finance*, 57(1), 369-403.

El-Erian, M.A. (2016). *The Only Game in Town – Central Banks, Instability, and Avoiding the Next Collapse.* (Random House, New York N.Y.)

Evstigneev, I. V., Hens, T., Schenk-Hoppé, K. R. (2008). Globally Evolutionarily Stable Portfolio Rules. *Journal of Economic Theory*, 140, 197-228.

Fliessbach, K., Weber, B., Trautner, P., Dohmen, T., Sunde, U., Elger, C.E., Falk, A. (2007). Social Comparison Affects Reward-Related Brain Activity in the Human Ventral Striatum. *Science*, 318 (23 November).

Gali, J. (1994). Keeping Up with the Joneses: Consumption Externalities, Portfolio Choice, and Asset Prices. *Journal of Money, Credit, and Banking*, 26(1), 1-8.

Gomez, J.P. (2007). The Impact of Keeping Up with the Joneses Behavior on Asset Prices and Portfolio Choice. *Finance Research Letters*, 4, 95-103.

Hsia, C.C. (1981). Coherence of the Modern Theories of Finance. *The Financial Review*, 16(1), 27-42.

Hu, J., Cao, Y., Blue, P.R., Zhou, X. (2014). Low Social Status Decreases the Neural Salience of Unfairness. *Frontiers in Behavioral Neuroscience*, 8.

Huang, C. and Litzenberger, R. (1988). *Foundations for Financial Economics*. (Prentice-Hall, Inc., Englewood Cliffs, New Jersey)

Jondeau, E. Rockinger, M. (2006). Optimal Portfolio Allocation under Higher Moments. *European Financial Management*, 12(1) (2006), 29-55.

Kelley, J. L. JR. (1956). A New Interpretation of Information Rate. *The Bell System Technical Journal*, (July).

Kraus, A. and Litzenberger, R.H. (1976). Skewness Preference and the Valuation of Risk Assets. *Journal of Finance*, 31, 1085-1100.

Lillo, F. and Mantegna, R.N. (2000). Symmetry Alteration of Ensemble return Distribution in Crash and Rally Days of Financial Markets. *The European Physical Journal B*, 15, 603-606.

Long, J.B. (1990). The Numeraire Portfolio. *Journal of Financial Economics*, 26, 29-69.

Markowitz. H. (1952). Portfolio Selection. *Journal of Finance*, 7(1), 77-91.

Merton, R.C. (1980). On Estimating the Expected Return on the Market- An Exploratory Investigation. *Journal of Financial Economics*, 8, 323-361.

Piketty, T. (2014). *Capital in the Twenty-First Century*, in Goldhammer, A. (Trans.). (The Belknap Press of Harvard University, Cambridge, Massachusetts)

Pownall, R.A.J. and Koedijk, K.G. (1999). Capturing Downside Risk in Financial Markets: The Case of The Asian Crisis. *Journal of International Money and Finance*, 18, 853-870.

Scott, R.C. and Horvath, P. A. (1980). On the Direction of Preference for Moments of Higher Order than the Variance. *Journal of Finance*, 35, 915-919.

Sharpe, W.F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance*, 19(3), 425-442.

Sinn, H.W. and Weichenrieder, A.J. (1993). Biological Selection of Risk Preferences. in *Risk is a Construct– Perceptions of Risk Perception*. (Munich: Knesebeck).

Swencionis, J.K. and Fiske, S.T. (2014). How Social Neuroscience Can Inform Theories of Social Comparison. *Neuropsychologia*, 56, 140-146.

Tricomi, E., Rangel, A., Camerer, C.F., O'Doherty, J.P. (2010). Neural Evidence for Inequality-Averse Social Preferences. *Nature*, 463, (25 February), 1089-1092.

Zink, C.F., Tong, Y., Chen, Q., Bassett, D.S., Stein, J.L., Meyer-Lindenberg, A. (2008). Know Your Place: Neural Processing of Social Hierarchy in Humans. *Neuron*, 58, 273-283.

Appendix A

This appendix provides an example that invokes a simple economy to illustrate the mathematics of the model.

Assume there are two assets, Y and Z, in the economy. The weight of asset Y in the market portfolio is 0.5 and that of asset Z is 0.5. The following table shows the probability distributions (equal probabilities) of the simple gross returns of assets Y, Z, and the market portfolio.

Asset Y	Asset Z	Market Portfolio
0.6	0.75	0.675
0.9	0.8	0.85
1.1	0.9	1
1.25	0.95	1.1
1.35	1.2	1.275
1.45	1.55	1.5
1.5	2.295	1.8977
Mean = 1.1643 Std. Dev. = 0.3	Mean = 1.2065 Std. Dev. = 0.5123	Mean = 1.1854 Std. Dev. = 0.3834

Note that the first cell in the column for asset Y and the last cell in the column for asset Z where adjusted so that the economy satisfies equation (21). The return distribution for asset Y is skewed to the left. That for asset Z is skewed to the right. The following results can be easily obtained.

$$E\left[\frac{1}{1+\tilde{R}_m}\right]=0.935 \quad r_f=0.0695$$

$$E\left[\tilde{R}_Y\right]-r_f=0.0948 \quad E\left[\tilde{R}_Z\right]-r_f=0.137 \quad E\left[\tilde{R}_m\right]-r_f=0.1159$$

$$\frac{E\left[\tilde{R}_Y\right]-r_f}{E\left[\tilde{R}_m\right]-r_f}=0.818 \quad \frac{E\left[\tilde{R}_Z\right]-r_f}{E\left[\tilde{R}_m\right]-r_f}=1.182$$

The chi and beta factors for both assets are as follows:

$$\chi_{Ym}=0.818 \quad \beta_{Ym}=0.6568$$

$$\chi_{Zm}=1.182 \quad \beta_{Zm}=1.2936$$

Appendix B

Equation (3) is restated below:

$$E\left[\frac{\partial U^i}{\partial RW_1^i}\frac{\left(\tilde{P}_{1j}-P_{0j}\left(1+r_f\right)\right)}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]=0 \quad \forall j$$

From the definition of covariance:

$$E\left[\frac{\partial U^i}{\partial RW_1^i}\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]E\left[\tilde{P}_{1j}-P_{0j}\left(1+r_f\right)\right]=-Cov\left(\frac{\partial U^i}{\partial RW_1^i}\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}},\left(\tilde{P}_{1j}-P_{0j}\left(1+r_f\right)\right)\right)$$

$$\left(E\left[\frac{\partial U^i}{\partial RW_1^i}\right]E\left[\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]+Cov\left(\frac{\partial U^i}{\partial RW_1^i},\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right)\right)E\left[\tilde{P}_{1j}-P_{0j}\left(1+r_f\right)\right]=-Cov\left(\frac{\partial U^i}{\partial RW_1^i}\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}},\tilde{P}_{1j}\right)$$

Assuming quadratic utility:

$$\left(E\left[a^i-A^iRW_1^i\right]E\left[\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]-A^i\,Cov\left(RW_1^i,\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right)\right)E\left[\tilde{P}_{1j}-P_{0j}\left(1+r_f\right)\right]=$$

$$-a^iCov\left(\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}},\tilde{P}_{1j}\right)+A^iCov\left(\frac{RW_1^i}{\sum_{j=1}^{n}\tilde{P}_{1j}},\tilde{P}_{1j}\right)$$

Divide both sides by A^i :

$$\left(\left[\varphi^i\right]^{-1}E\left[\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]-Cov\left(RW_1^i,\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right)\right)E\left[\tilde{P}_{1j}-P_{0j}\left(1+r_f\right)\right]=$$

$$-\frac{a^i}{A^i}Cov\left(\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}},\tilde{P}_{1j}\right)+Cov\left(\frac{RW_1^i}{\sum_{j=1}^{n}\tilde{P}_{1j}},\tilde{P}_{1j}\right)$$

Aggregating across all *n* agents:

$$\left[\sum_{i=1}^{m}\varphi^{i^{-1}}\right]E\left[\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}}\right]E\left[\tilde{P}_{1j}-P_{0j}\left(1+r_f\right)\right]=$$

$$-Cov\left(\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}},\tilde{P}_{1j}\right)\left(\sum_{i=1}^{m}\frac{a^i}{A^i}\right)+Cov\left(\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}},\tilde{P}_{1j}\right)$$

$$=Cov\left(\frac{1}{\sum_{j=1}^{n}\tilde{P}_{1j}},\tilde{P}_{1j}\right)\left(-\sum_{i=1}^{m}\frac{a^i}{A^i}+1\right)$$

$$E\left[\tilde{P}_{1j} - P_{0j}\left(1+r_f\right)\right] = \left[\sum_{i=1}^{m} \varphi^{i\ -1}\right]^{-1} \left(-\sum_{i=1}^{m} \frac{a^i}{A^i} + 1\right) \frac{Cov\left(\dfrac{1}{\sum_{j=1}^{n} \tilde{P}_{1j}}, \tilde{P}_{1j}\right)}{E\left[\dfrac{1}{\sum_{j=1}^{n} \tilde{P}_{1j}}\right]}$$

Dividing both sides by P_{0j} and multiplying R.H.S. by $\dfrac{\sum_{j=1}^{n} P_{0j}}{\sum_{j=1}^{n} P_{0j}}$

$$E\left[\tilde{R}_j - r_f\right] = \left[\sum_{i=1}^{m} \varphi^{i\ -1}\right]^{-1} \left(-\sum_{i=1}^{m} \frac{a^i}{A^i} + 1\right) \frac{Cov\left(\dfrac{1}{1+\tilde{R}_m}, \tilde{R}_j\right)}{E\left[\dfrac{1}{1+\tilde{R}_m}\right]}$$

Thus equation (10) is obtained since $\sum_{i=1}^{m} \varphi^{i\ -1} = \left(\sum_{i=1}^{m} \dfrac{a^i}{A^i} - 1\right)$

Next assume that there is a single risky asset and one risk-free asset. The minimum risk premium needed to induce an agent to invest all his wealth in the risky asset[11] can be found from the following relationship which is a direct result of equation (3) applied to this special case

$E\left[\dfrac{\partial U^i}{\partial RW_1^i}\left(\dfrac{W_0^i\left(1+\tilde{R}\right)}{V_0\left(1+\tilde{R}\right)}\right)\dfrac{\left(\tilde{R}-r_f\right)}{V_0\left(1+\tilde{R}\right)}\right] \geq 0$ where W_0^i is the invested wealth of agent *i*.

Since there is no uncertainty about the (positive) marginal utility

$E\left[\dfrac{\left(\tilde{R}-r_f\right)}{\left(1+\tilde{R}\right)}\right] \geq 0$ and using the definition of covariance

$$E\left[\tilde{R} - r_f\right] \geq \frac{-Cov\left(\tilde{R}, \dfrac{1}{\left(1+\tilde{R}\right)}\right)}{E\left[\dfrac{1}{\left(1+\tilde{R}\right)}\right]}$$

This is similar to equation (10). An intuitive explanation for the disappearance of the risk aversion factor is included in section II.

[11] The development here follows steps similar to Huang and Litzenberger (1988).

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Appendix C

As stated in section II, the RWCAPM is based on agents having a utility function of the form $U = f(E_W, -\text{cov}(\tilde{W}, \frac{1}{\tilde{W}})$. This appendix utilizes this form of the utility function in an analysis that provides further support to the argument that the RWCAPM is an equilibrium model. The analysis is based on a methodology that is used by some authors to derive the basic CAPM, e.g. Copeland and Weston (1992, pp. 195-198).

The methodology takes as a starting point a portfolio "…consisting of *a* % invested in risky asset *j* and (1-*a*) % in the market portfolio," and derives expressions for the expected return and standard deviation for such a portfolio. The derivative of each of these expressions with respect to (w.r.t.) *a* is found. ***a*** represents the excess demand for asset *j* which has to equal zero under the basic CAPM for any individual investor because all investors hold the market portfolio. The ratio of the expression for the derivative of the portfolio's expected return w.r.t. *a* to the expression for the derivative of the portfolio's standard deviation of returns w.r.t. *a* is specified at *a* = zero. This is taken to represent "The slope of the risk-return trade-off at point *M*," the market portfolio on the portfolio frontier. Adding the insight that this slope "…must also be equal to the slope of the capital market line," leads after some algebraic manipulations to the basic CAPM.

Under the RWCAPM investors do not hold identical portfolios and ***a*** need not be equal to zero for individual investors; rather equilibrium would be manifested in that it's weighted average across investors is zero. However, to be able to apply the abovementioned methodology, it is assumed that ***a*** is roughly equal to zero for individual investors. The purpose is not to derive the RWCAPM (which is derived in section II) but to support the argument that the RWCAPM is an equilibrium structure.

$$\textit{Portfolio Expected return}$$
$$\overline{R}_P = a\overline{R}_j + (1-a)\overline{R}_m$$
$$\textit{Risk Measure}: -Cov\left(1 + a\tilde{R}_j + (1-a)\tilde{R}_m, \frac{1}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right) =$$
$$-a\,Cov\left(\tilde{R}_j, \frac{1}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right) - (1-a)Cov\left(\tilde{R}_m, \frac{1}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right) =$$
$$-aE\left[\frac{\tilde{R}_j - \tilde{R}_m}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right] - E\left[\frac{\tilde{R}_m}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right] +$$
$$aE\left[\tilde{R}_j - \tilde{R}_m\right]E\left[\frac{1}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right] + \overline{R}_m E\left[\frac{1}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right]$$

Rj and Rm are the returns of asset j and the market portfolio respectively.

The derivative of the portfolio's expected return w.r.t. a is:

$$\frac{\partial \overline{R}_P}{\partial a} = \overline{R}_j - \overline{R}_m$$

The derivative of the portfolio's risk measure w.r.t. *a* is:

$$\frac{\partial - Cov\left(1 + a\tilde{R}_j + (1-a)\tilde{R}_m, \frac{1}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right)}{\partial a} =$$

$$-Cov\left(\tilde{R}_j - \tilde{R}_m, \frac{1}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right) + a\,Cov\left(\tilde{R}_j - \tilde{R}_m, \frac{\tilde{R}_j - \tilde{R}_m}{\left(1 + a\tilde{R}_j + (1-a)\tilde{R}_m\right)^2}\right) +$$

$$Cov\left(\tilde{R}_m, \frac{\tilde{R}_j - \tilde{R}_m}{\left(1 + a\tilde{R}_j + (1-a)\tilde{R}_m\right)^2}\right)$$

Specifying the ratio of the expression for the derivative of the portfolio's expected return w.r.t. *a* to the expression for the derivative of the portfolio's risk measure w.r.t. *a*, at *a* = zero leads to:

$$\frac{\partial \bar{R}_P}{\partial a} \Bigg/ \frac{\partial - Cov\left(1 + a\tilde{R}_j + (1-a)\tilde{R}_m, \frac{1}{1 + a\tilde{R}_j + (1-a)\tilde{R}_m}\right)}{\partial a}_{@\,a=zero} \tag{C-1}$$

$$= \frac{\bar{R}_j - \bar{R}_m}{-Cov\left(\tilde{R}_j, \frac{1}{1+\tilde{R}_m}\right) + Cov\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right) + Cov\left(\tilde{R}_m, \frac{\tilde{R}_j - \tilde{R}_m}{\left(1+\tilde{R}_m\right)^2}\right)}$$

This represents the slope of the risk-return trade-off at the market portfolio on a portfolio frontier.

The slope of the capital market line (CML) drawn as a tangent to this portfolio frontier from the risk-free rate is given by:

$$\frac{\bar{R}_m - r_f}{-Cov\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right)} \tag{C-2}$$

Equating the right hand side of equation (C-1) to the slope of the CML in (C-2) leads to the following:

$$E\left[\tilde{R}_j\right] - r_f = \chi_{jm}\left(E\left[\tilde{R}_m\right] - r_f\right) - \frac{Cov\left(\tilde{R}_m, \frac{\tilde{R}_j - \tilde{R}_m}{\left(1+\tilde{R}_m\right)^2}\right)}{Cov\left(\tilde{R}_m, \frac{1}{1+\tilde{R}_m}\right)}\left(E\left[\tilde{R}_m\right] - r_f\right) \tag{C-3}$$

This equation deviates from the RWCAPM due to the second term on the right hand side (which vanishes when $j=m$). Nevertheless the RWCAPM is clearly represented through the first term. The deviation of equation (C-3) from the RWCAPM might be due to the assumption that ***a*** = zero for individual investors; this assumption is applicable in the case of the basic CAPM but is not accurate in the case of the RWCAPM where it is the weighted average of ***a*** across investors that is zero.

It is to be noted also that in deriving the basic CAPM using this methodology the risk measure used is the standard deviation rather than the variance; if the variance is used extraneous terms appear that cloud the basic CAPM.