Shemetov, Valery. "Probability of Default in Corporate Economic Distress, or What Risk Does Market Reward?." *ACRN Oxford Journal of Finance and Risk Perspectives 5.1 (2016): 183-205.*

PROBABILITY OF DEFAULT IN CORPORATE ECONOMIC DISTRESS, OR WHAT RISK DOES MARKET REWARD?

VALERY SHEMETOV [1]
[1]Northern Virginia Community College

***Abstract**: The article suggests a quantitative model describing development of a corporate economic distress when a firm is not burdened with a long-term debt yet. The model introduces new variables related to the crisis dynamics, market trend and volatility, and corporate features. For the economic distress left unattended and for the recovery stage when the firm tries to restore its stability, the probability of default as a function of time and problem parameters is given, and the distance to default and the point of no return for launching a recovery program are estimated. The model helps select the program minimizing the probability of default over a set of available recovery programs. For a steady developing corporation, it is estimated how much money can be withdrawn from the business for dividend payments and other needs without exposing the corporation to an extra risk of default. In the approximation of firms having no long-term debt, the model demonstrates the limits of validity of the Capital Assets Pricing Model. (JEL G30)*

***Keywords**: Corporate Economic Distress; Recovery Program; Probability of Default; Distance to Default; Sustainable Corporate Development; CAPM*

Introduction

In this paper we consider development of a corporate economic distress which is usually the first stage of a more general and dangerous phenomenon of the financial distress (see, for example, Altman & Hotchkiss (2006), Asquith, Gertner, et.al. (1994), Bibeault (1982), Gordon (1971), Wruck (1990)). Following Gorbenko & Strebulaev (2010), we interpret the corporate distress as a permanent shock generated by an external/internal corporate event and setting a long-term adverse factor in a corporate business environment that makes the corporate return on assets (ROA) decrease over time. In a volatile business environment, the first stage of a crisis develops imperceptibly for the company during an incubation period making the company loose precious time. If the corporation fails to identify the problem and find an adequate response to it, the company sustains ever-increasing losses whose cumulative effect threatens a corporate market position and can lead to default. A good recovery plan launched promptly and implemented strictly to the schedule minimizes possible damages and has a good chance to restore corporate stability. The question is *what characteristics the recovery plan must have to be good, and how they are related to parameters of the corporate business environment*.

Crisis management focuses on the following issues:

1. *Crisis recognition*, *i.e.* registration of a mismatch between a current corporate strategy and a new environment trend germinated by external/internal events. Weak signals of the newborn crisis are concealed by fluctuations in the corporate environment; so, a company must have an early diagnostics procedure for detecting those signals as soon as possible.
2. *Crisis identification*. The management team reviews the company's business to diagnose the crisis, localize its causes, study their mechanisms, and estimate possible threats and opportunities (Pearson & Claire, 1998).
3. *Development and implementation of a recovery program*. Any recovery program must meet specific challenges of a particular business and industry pattern, and, therefore, cannot be comprehensively described here. However, all programs have the following in common. The objective of a recovery program is getting a crisis under control with a minimal loss and probability of default. Decision-making occurs under pressure of losses, uncertainty, and time deficit. Because of the uncertainty, only probabilistic estimations of a corporate development are possible. Any recovery program starts with a delay about the crisis onset and has some efficiency in restoring corporate ROA. For a set of recovery programs available, the team selects the best one and takes steps to its implementation such as fundraising, making necessary organizational and operational preparations, etc. When ready, the corporate team starts implementation of the chosen plan trying to restore a safe business position.
4. *Post-crisis management*. If the company succeeds, the new strategy outlined in the crisis gets further development carrying the changes in organizational values, mission, structure, policies, business processes to their logical conclusions. We share the Roux-Dufort's position (2007) that this holistic approach can create a company's new strategic position in the market increasing its long-term survivability.

In a crisis the corporate team meets a challenge: they must solve the problem which is quite new and maybe even ill-structured in the current paradigm, and they must do it fast. But how fast the corporate reaction must be, and how parameters of the company, business environment, crisis, and recovery program affect the probability of survival the crisis, these questions still wait for their answers. This paper tries to answer them.

For more than fifty years of intensive study of financial distress, a lot of techniques have been developed predicting corporate default based on the multivariate discriminant analysis (Altman, 1968), logit and probit analysis (Asquith, Gertner, et.al., 1994), cumulative sums methodology (Kahya & Theodossiou, 1999)), neural networks (Salchenberger, Cinar, et.al., 1992), genetic algorithms (Shin & Lee, 2002) and some others. All those methods belong to a set of classifying algorithms seeking for time-independent criteria for attributing a firm under consideration to a cluster of healthy firms or to a cluster of distressed ones. They do not consider dynamics of a corporate crisis and efforts the company undertakes to survive it. Therefore, these methods cannot calculate the probability of default as a function of time and industry, company, crisis, and recovery program parameters.

Structural models estimating the probability of corporate default as a function of time from the creditor's point of view are considered in many papers starting with the Merton's seminal work (1974) (e.g. Black & Cox (1976), Geske (1977), Lando (1998)). All those structural models use the option pricing model developed by Black & Scholes (1972) and consider development of a default at a financially distressed firm as a Markov process (a process without prehistory). Financially distressed companies, however, do have a prehistory due to their accumulated long-term debt. Therefore, strictly speaking, structural models describe the development of default of economically distressed firms having no long-term debt. The main difference of this paper from "the creditor's approach" to the problem of corporate default is that it considers a corporate distress from the inside using information which help the team not only estimate the distance to default, but also choose a recovery program providing for the highest probability of survival among all available programs.

Outecheva (2007) determines a corporate default as an event occurring when (i) a company fails to meet its financial obligations, (ii) a company files bankruptcy, (iii) an exchange is distressed. This paper considers just the first two cases when a company becomes unable to pay its obligatory payments, or files bankruptcy.

The following analysis is based on the assumptions:

A1) corporate ROA changes uniformly in a crisis

A2) market fluctuations are normally distributed, time-invariant, and delta-correlated with slowly changing (constant at the interval of the distress development) parameters

A3) a rate of corporate obligatory payments is a piecewise constant function.

Assumption (A1) reflects our approximate description of a crisis. Of course, decrease in corporate ROA can vary during the crisis making corporate managers adjust a chosen recovery program to changing conditions. So, an anti-crisis program is not a one-time act, but a controlling procedure reacting to the signals coming from the corporate business environment. Assumptions (A2) and (A3) are discussed in Section 1.

The remainder part of the paper is organized as follows. Section 1 describes the model, considers development of corporate economic distress left unattended up to the moment of default, and estimates the distance to default. Section 2 contains an analysis of a corporate recovery in a distress and estimates the probability of default during the recovery, the point of no return, and the maximum tolerable delay for a recovery program securing a given recovery rate of corporate ROA. Section 3 considers conditions for a steady corporate development and estimates the effect of taxes and a dividend policy on corporate long-term survival. Section 4 estimates the limits for changes in market risks and ROA that generate more free (unbounded) money for company needs and dividend payments. The last section contains a brief discussion of results and conclusions.

Model Description

A corporate strategy includes many components: strategic, financial, and organizational managements, marketing, etc. Its success results in a steady increase of corporate assets over a long-term period. For continuous time, asset dynamics $x(t)$ can be symbolically presented by the model:

$$\frac{dx}{dt} = [r(t) + n_0(t)]x - p(t), \quad x(0) = x_0 . \tag{1.1}$$

Function $r(t)$ is corporate return on assets (ROA) at time t, $p(t)$ is a rate of corporate obligatory payments at time t including fixed operating costs, long-term debts, R&D expenditures, marketing expenses, and all kinds of overheads like non-profile assets, etc. We suppose $p(t)$ to be a piecewise constant function:

$$p(t) = \begin{cases} p_0, \ 0 \le t < t_1, \\ p_1, \ t_1 \le t < t_2, \\ \ldots \\ p_i, \ t_i \le t < t_{i+1}, \\ \ldots \end{cases} \tag{1.2}$$

here p_i is a rate of corporate obligatory payment at interval $[t_i, t_{i+1})$. So, a company's state at this interval is affected by continuing debt payments, investments into R & D projects, and also current obligatory payments like fixed costs. Term $n_0\,(t)$ shows an effect of market fluctuations on corporate assets; it is supposed to be a normal random process with the properties:

(A2a) $\langle n_0(t) \rangle = m$;

(A2b) $n_0\,(t)$ is time-invariant and delta–correlated: $\langle [n_0(t_1) - m][n_0(t_2) - m] \rangle = C\delta(t_1 - t_2)$,

$\delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \ne t_0 \end{cases}$, and $\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0) dt = 1$ for any $\varepsilon > 0$.

Constant C is a measure of the effect that market fluctuations make on corporate assets. Parameter C reflects a balance between the fluctuations intensity and the company's capability to run her business in rough conditions limiting an adverse effect of market fluctuations on corporate assets. Operator $\langle \cdot \rangle$ denotes averaging over $n_0\,(t)$ at time t. Assumption (A2b) means that a great number of non-correlated random fluctuations occur in a characteristic period of ROA change. Assumption (A2a) actually reflects the effect that the market exercises on any company. Process $n_0(t)$ includes all random market impacts on a company, both diversifiable (unique risks) and non-diversifiable ones (market risks). The market risks contain a drift with the market that means the presence of a non-zero regular component in $n_0\,(t)$ for time periods: $m < 0$ for recessive markets, $m = 0$ for stagnating markets, and $m > 0$ for raising markets.

With a process $n(t)$: $n_0(t) = n(t) + m$ which is normal, time-invariant, delta-correlated, and has a zero mean, equation (1.1) becomes

$$\frac{dx}{dt} = [r(t) + m + n(t)]x - p(t), \quad x(0) = x_0, \tag{1.3}$$

which can be rewritten as the geometric Brownian model

$$dx = [r_0(t)x - p(t)]dt + \sqrt{C}x\delta W, \quad x(0) = x_0, \quad r_0(t) = r(t) + m, \tag{1.4}$$

where W is a Wiener process. For the last stochastic equation one can write a Fokker - Planck equation for the probability distribution $f(x,t)$

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x}\left[(r_0(t)x-p)f\right]-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(Cx^2 f\right)=0\,. \tag{1.5}$$

Introducing a new function V and a new variable z

$$V = xf(x,t), \quad z=\ln x, \tag{1.6}$$

the equation (1.5) becomes

$$\frac{\partial V}{\partial t}+r_1(t)\frac{\partial V}{\partial z}-\frac{C}{2}\frac{\partial^2 V}{\partial z^2}+pe^{-z}\left(V-\frac{\partial V}{\partial z}\right)=0\,, \quad r_1(t)=r(t)+m-C/2 \tag{1.7}$$

In this paper we consider the marginal case of $p=0$, described by the equation

$$\frac{\partial W}{\partial t}+r_1(t)\frac{\partial W}{\partial z}-\frac{C}{2}\frac{\partial^2 W}{\partial z^2}=0 \tag{1.8}$$

with the initial condition

$$W(z,0)=\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp\left\{-\frac{(z-H_0)^2}{2\sigma_0^2}\right\}, \quad H_0=\langle z\rangle|_{t=0}\,. \tag{1.8a}$$

This problem has a solution

$$W(z,t)=\frac{1}{\sqrt{2\pi(\sigma_0^2+Ct)}}\exp\left\{-\frac{(z-H(t))^2}{2(\sigma_0^2+Ct)}\right\}, \tag{1.9a}$$

$$H(t)=H_0+\int_0^t r_1(\tau)d\tau \tag{1.9b}$$

which is a normal distribution parametrically depending on time. Equation (1.9b) shows an evolution of the distribution center (the distribution mean) over time. One can see from (1.7) that function $V(z,t)$ deviates from the normal distribution the more the larger parameter *p*. Solution (1.9) supposes that corporate assets can assume any value from zero to infinity. However, there is a minimal value $x_{\min}>0$, $DL=\ln x_{\min}$, at which default occurs. For a small business (a proprietorship), $x_{\min}$ is an asset value at which the proprietors stop their struggle for the firm survival and file bankruptcy trying to confine damages to their personal property. For a public firm, $x_{\min}$ is a level at which a noteworthy part of corporate shareholders panicking sells their shares out in a short time interval making the share price drop sharp. Even if that drop does not lead to a corporate default immediately, the problem of corporate survival must be restated for the new (obviously worse) conditions. Observe that x_{min} depends on both objective factors (a value of the accumulated debt, the rate of the assets decrease, etc.) and subjective factors (the management averse to risk, shareholders' confidence in the corporate team, etc.). Subjective factors add more ambiguity to the problem of corporate survival.

For the problem (P1) described by equation (1.8) with the boundary conditions

1. There is an absorbing screen at the default line : $W(DL,t)=0$
2. $W(z,t)$ decays fast enough as *z* tends to infinity: $W(\infty,t)=0$

and the initial condition meeting the boundary conditions

$$W(z,0)=\frac{1}{\sqrt{2\pi\sigma_0^2}}\left\{\exp\left[-\frac{(z-H_0)^2}{2\sigma_0^2}\right]-\exp\left[-\frac{(-2DL+z+H_0)^2}{2\sigma_0^2}\right]\right\}, \qquad (1.8b)$$

a solution is

$$W(z,t)=\frac{1}{\sqrt{2\pi(\sigma_0^2+Ct)}}\left\{\exp\left[-\frac{(z-H(t))^2}{2(\sigma_0^2+Ct)}\right]-\exp\left[-\frac{(-2DL+z+H(t))^2}{2(\sigma_0^2+Ct)}\right]\right\}. \qquad (1.S)$$

The physical meaning of the absorbing screen becomes clear if one considers Brownian particles whose concentration is proportional to $W(z,t)$ travelling in the semi-space of [*DL*, ∞). When a particle touches the screen, it sticks to the screen (the particle "perishes"). To see the financial meaning of that screen, let us find the intensity of the probability to perish for the particles with the probability distributions determined by the problem (P1):

$$P_D[H(t)]=2\int_0^{x_{min}}\frac{dx}{x\sqrt{2\pi(\sigma_0^2+Ct)}}\exp\left\{-\frac{(\ln x-H(t))^2}{2(\sigma_0^2+Ct)}\right\}=2\Phi\left(-\frac{H(t)-DL}{\sqrt{\sigma_0^2+Ct}}\right), \qquad (1.10a)$$

$$DL=\ln x_{min}\,, \qquad \Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp(-t^2/2)\,dt\,. \qquad (1.10b)$$

At the screen $H(t)=DL$, and the intensity of the particle's probability to perish is $P_D(DL)=1$. The case when the mean logarithm of assets hit *DL*-line, we call the *hard default*. The other case when a default occurs because of fluctuations while the mean logarithm of assets stays over *DL*-line, we call the *soft default*. The probability of the soft default, PRD, as the assets decline is determined as

$$PRD(t)=\int_0^t P_D[H(t')]dt'\,. \qquad (1.11)$$

Suppose that a corporation develops with a constant ROA:

$$r_1(t)\equiv R_1=R+m-C/2\,, \qquad (1.12)$$

and average corporate assets grow exponentially over time. Here R_1 is observable corporate ROA in current market conditions, and R is unobservable ROA in "sterile" conditions with $m = 0$, $C = 0$.

Let a crisis onset at moment T_A and since that time the mean corporate ROA uniformly decreases over time:

$$r_1(t) = R_1 - wt\,, \tag{1.13}$$

where w is a rate of ROA decrease caused by the crisis (Fig. 1). Values T_A and w can be determined using corporate statistics collected due to monitoring of corporate assets. Positive asset dynamics together with an effective asset structure determine a corporate long-term stability. If the asset structure is not optimal for a current business conditions, in other words some assets cannot be used, one must use in the analysis effective assets $x_{eff} = \alpha x(t),\ \alpha < 1$.

From the equation $P_D(t) = 1$, time T_{Ch} when the mean logarithm of assets hits *DL*-line (the distance to the hard default) is

$$T_{Ch} = T_B + \sqrt{2(H_B - DL)/w}\,, \tag{1.14a}$$

$$T_B = R_1 / w\,, \quad H_B = H(T_B) = H_A + wT_B^2/2\,. \tag{1.14b}$$

T_B is the moment when corporate assets becomes maximum, and H_B is the highest point in the trajectory of corporate decline.

The distance to a soft default is given by the equation

$$\Delta T^2 = (2/w)[H_B - DL - \alpha_p(\sigma_0^2 + CT_B + C\Delta T)^{1/2}]\,, \tag{1.15.a}$$

$$T_{Cs}(p_0) = T_B + \Delta T\,, \qquad \alpha_p = -\Phi^{-1}(p_0/2) > 0\,, \tag{1.15.b}$$

where p_0 is a given intensity of default considered as dangerous. The equation can be solved numerically using iterative procedure starting with $\Delta T^0 = 0$ in the right part of the equation. As one can easily see, $T_{Cs}(p_0) < T_{Ch}$ and tends to T_{Ch}, as $p_0 \to 1$.

Corporate recovery

Let a recovery program be launched at time T_1 raising corporate ROA at rate u. At T_1, the recovery initial conditions are:

$$\begin{aligned} &r_1(T_1) = -w(T_1 - T_B), \\ &H_1 = H(T_1) = H_A + wT_BT_1 - wT_1^2/2 = H_B - w(T_1 - T_B)^2/2, \\ &\sigma_1^2 = \sigma_0^2 + CT_1. \end{aligned} \qquad (2.1)$$

So, for t measured from T_1, we have in the recovery

$$r_2(t) = ut - w(T_1 - T_B) \ ,$$

$$H(t) = H_1 - w(T_1 - T_B)t + ut^2/2 \equiv H_2(t),$$

$$(2.2)$$

$$H(t) = \begin{cases} H_1(t) = H_A + wT_Bt - wt^2/2, & 0 \le t \le T_1 \\ H_2(t') = H_1 - w(T_1 - T_B)t' + ut'^2/2, & t' = t - T_1,\ t > T_1 \end{cases}$$

If $T_1 > T_B$, a critical point exists in the recovery trajectory in which a mean logarithm of corporate assets is minimal:

$$T_{cr} = (w/u)(T_1 - T_B), \qquad H_{cr} = H_1 - (w^2/2u)(T_1 - T_B)^2 = H_1 - uT_{cr}^2/2. \qquad (2.3)$$

The intensity of the probability of default in the recovery when a mean logarithm of assets is $H(t)$ can be computed as

$$P_D(H) = 2\Phi\left[-\frac{H(t) - DL}{\sqrt{\sigma_1^2 + Ct}}\right], \qquad (2.4)$$

and the probability of corporate default is determined by the integral

$$PRD(t; T_1, H_1, u) = \int_{T_1}^{t} P_D[H(t')]dt'. \qquad (2.5)$$

It becomes clear from (2.2), (2.4) and (2.5) that the main contribution to the probability of soft default is made by the part of the recovery trajectory closest to the critical point where the intensity of the probability of default is high and the vertical velocity is low; the time that the mean logarithm of assets spends in the dangerous area depends inversely on recovery rate *u*.

If the objective of the recovery program is to restore the pre-crisis ROA: $r_2(T_r) = R_1$, then one can find the recovery time T_r and mean logarithm of assets H_r as

$$T_r = (w/u)T_1, \quad H_r = H_1 + (w^2/2u)(2T_B - T_1)T_1 \tag{2.6}$$

For $T_1 > 2T_B$, however, $H_r < H_1$. If H_r is too close to *DL*-line, and the intensity of the default probability remains intolerably high then the objective of the recovery can be redefined as to achieve a safe state where the intensity of the probability of corporate default is less than a predetermined value p_0.

Now recovery time T_r is determined by the equations

$$T_r = T_{cr} + x, \quad x^2 = (2/u)\left(\alpha_p \sqrt{\sigma_{cr}^2 + Cx} - H_{cr} + DL\right), \tag{2.7a}$$

$$\sigma_{cr}^2 \equiv \sigma_1^2 + CT_{cr}, \qquad \Phi^{-1}(p_0/2) \equiv -\alpha_p < 0 \tag{2.7b}$$

here $\Phi^{-1}(\cdot)$ is the inverse function for $\Phi(z)$.

The *marginal delay* with a launch of the recovery program, $T_1^{\max}$, for which the hard default still occurs, can be found as a time for which the numerator in Eq. (2.4) turns zero at the critical point, that is

$$H_{cr} - DL = 0.$$

This condition has two consequences. For the program with a fixed expected recovery rate *u*, the marginal delay for its launch $T_1^{\max}(u)$ is

$$T_1^{\max} = T_B + \sqrt{\frac{u}{u+w}}(T_{Ch} - T_B), \tag{2.8}$$

and for any $T_1 < T_1^{\max}$ there is no hard default in the recovery. For the program with a fixed delay T_1, the minimal rate $u^{\min}(T_1)$ preventing the hard default in the recovery is

$$u^{\min} = w\frac{H_B - H_1}{H_1 - DL}. \tag{2.9}$$

For management purposes, however, it is more practical to know the *maximum tolerable delay* for chosen p_0 for the maximum intensity of the probability of default $T_{1p} \equiv T_1(P_D^{\max} = p_0)$ which is a solution to the equation:

$$T_{1p} = T_B + \Delta T\,, \quad \alpha_p = -\Phi^{-1}(p_0/2)\,,$$

$$\Delta T^2 = \frac{2u}{w(u+w)}\left(H_B - DL - \alpha_p \sqrt{\sigma_0^2 + \frac{Cw}{u}(T_B + \Delta T)} \right). \qquad (2.10)$$

The maximum tolerable delay T_{1p} determines the *timeliness* of the anti-crisis decision for given environment, crisis, corporate, and recovery program's parameters and a chosen intensity of the default probability p_0; for any delay $T_1 < T_{1p}$ the maximum intensity of the probability of default remains less than p_0. In the same way, Eq. (2.10) determines the minimal recovery rate $u_{p_0}(T_1) = u^{\min}(P_D^{\max} = p_0)$ providing a chosen level of the maximum intensity of the probability of corporate default for a given delay T_1.

The usual corporate reaction to a possible default is to increase corporate assets borrowing money from banks (the small enterprises' tactics) or from the market as large firms do. However, this leads to regular debt payments that is the rate of payments becomes non-zero, $p(t) > 0$. We consider this case in another study.

Conditions for steady corporate progress

Let us return to a firm whose assets increase over time with a constant ROA, R_1. Because a standard deviation of corporate assets continuously grows over time, a question arises about the conditions providing for a steady corporate development. The intensity of the probability of default for that corporation is

$$P_D(H) = 2\Phi\left[-\frac{H(t) - DL}{\sigma_0\sqrt{1 + Ct/\sigma_0^2}} \right], \quad H(t) = H_0 + R_1 t\,, \qquad (3.1)$$

where $H_0 = \langle z \rangle |_{t=0}$ and σ_0^2 are the mean logarithm and variance of the logarithm of corporate assets at the initial time. A corporate development is considered as steady one if at any time t the intensity of the probability of default remains less than a chosen p_0. This requirement is equivalent to

$$1 + \frac{Ct}{\sigma_0^2} - \frac{\alpha_p C}{R_1\sigma_0}\sqrt{1 + \frac{Ct}{\sigma_0^2}} + \frac{C(H_0 - DL)}{R_1\sigma_0^2} - 1 > 0\,, \qquad (3.2)$$

and $\alpha_p = -\Phi^{-1}(p_0/2)$. The inequality is true for any t when the following requirement is met:

$$\left(\frac{\alpha_p C}{2\sigma_0 R_1}\right)^2 - \frac{C(H_0 - DL)}{R_1\sigma_0^2} + 1 < 0\,. \quad (3.3)$$

So, for ratio $Q = \alpha_p C/(2R_1\sigma_0)$ providing a steady corporate progress, one gets the interval

$$K^{-1}(ms) < Q < K(ms)\,, \quad (3.4a)$$

$$K(ms) = ms + (ms^2 - 1)^{1/2}\,, \quad (3.4b)$$

$$ms = \frac{H_0 - DL}{\alpha_p\sigma_0} \geq 1\,. \quad (3.4c)$$

As one can see in Fig.3, *K*(*ms*) (the upper branch) monotonously grows over *ms* from 1 to infinity, and K^{-1} (the lower branch) tends from one to zero. The condition (3.4c) means that H_0 must be high enough over *DL*-line, or else a default can occur soon after the start of corporate activities. Thus, variable *ms* can be interpreted as the *margin of corporate safety*. The right limit in (3.4*a*) seems very natural: the mean logarithm of corporate assets must rise faster than the standard deviation keeping a low intensity of the probability of default. The left inequality in (3.5*a*) limits ROA from above. When ROA is high, corporate assets grow fast and asset fluctuations become also high because they are proportional to the assets (see (1.4). The greater fluctuations need the greater margin of safety, therefore on the lower branch (high ROA) *ms* increases fast.

Any business is created as a source of income for its proprietors and/or investors, and any firm (not a proprietorship) finishing a year with profits pays taxes. Suppose that a proprietorship has successfully completed its business year staying within the region of stability. As time runs, a position of a steady firm shifts right along *ms* axis due to accumulated assets (a mean logarithm of corporate assets in the end of the financial year is $H(T)$). How much money can the proprietors take out of their business without taking an extra risk of default in the next business cycle? Figure 3 answers this question.

First, parameter Q is calculated for corporate parameters and a chosen intensity of the probability of default p_0:

$$Q_0 = \alpha_p C/(R_1\sigma_0)\,, \qquad \alpha_p = -\Phi^{-1}(p_0/2)\,, \quad (3.5)$$

using for the standard deviation σ_0 the value achieved *in the end* of the business year. Then a horizontal line $Q = Q_0$ is drawn in Graph 3 to a point of intersection with curve $K(ms)$, and point $ms^*(Q_0)$ shows the least margin of safety for the next business cycle, that is the minimal initial value of the logarithm of assets guaranteeing the intensity of the probability of default less than p_0. The amount of money W which the proprietors can withdraw without increasing the risk of default in the next business cycle is

$$\ln W \le \alpha_p \sigma_0 [ms - ms^*(Q_0)], \quad (3.6)$$

here ms is the margin of safety in the end of the year. We can call W unbounded or free money understanding that this money is not bounded by the requirement of securing the corporate stability.

Now let us consider a public corporation paying taxes and dividends. If after tax corporate assets are less than the least margin of safety for given conditions $ms^*(Q_0)$

$$ms(T) \equiv [\ln[X_F - T(X_F - X_0)] - DL]/(\alpha_p \sigma_0) \quad (3.7a)$$

$$ms(T) < ms^*(Q_0) \quad (3.7b)$$

($X_F - X_0 > 0$ and T are before taxes corporate income and an effective tax rate, $ms(T)$ is the after tax margin of safety), the business is already under an elevated risk in the next cycle, and the investors cannot pay any dividends without increasing the risk of default.

If the after tax margin of safety $ms(T)$ is greater than the least possible margin of safety $ms^*(Q_0)$, the amount of dividends D that can be paid to the investors without increasing the risk of default in the next business cycle is

$$\ln D \le \alpha_p \sigma_0 [ms(T) - ms^*(Q_0)]. \quad (3.8)$$

Because the assets variance grows linearly over time, the margin of safety increases from year to year reducing the sum that proprietors/investors can withdraw from the business for their needs. Therefore the procedure of "general cleaning" $F(\sigma, t)$ in the organization is necessary on a regular basis (a cyclic procedure) to control its entropy level and bring the asset variance down to about the initial value. The corporate team must do it in parallel with a permanent struggle to keep parameter C as low as possible.

Solving problem (P1) in the next time interval with new initial data, one again gets the probability distribution for corporate assets as a composition of two lognormal distributions

$$f(x,t) = \frac{1}{x\sqrt{2\pi(\sigma_0^2 + Ct)}} \left\{ \exp\left[-\frac{(\ln x - H(t))^2}{2(\sigma_0^2 + Ct)} \right] - \exp\left[-\frac{(-2DL + \ln x + H(t))^2}{2(\sigma_0^2 + Ct)} \right] \right\}. \quad (3.9)$$

In the end of the next business cycle the proprietors repeat one-time money withdrawal from their business observing the conditions of the corporate steady development. The probability distribution will have the same structure (3.9) in a business cycle interval for any number of completed cycles. However, for a time interval much greater than a business cycle, the probability distribution averaged over that interval will deviate from the distribution (3.9). Due to regular money withdrawals, corporate assets spend more time in a region of low values in comparison to the distribution (3.9) swelling the low part of the effective probability distribution. To determine that true probability distribution with heavier tails, one has to consider the case with nonzero payments $p(t) \neq 0$.

What risks does the market reward?

There is a popular statement made within the frames of Capital Asset Pricing Model (CAPM) that "the market rewards market risks" meaning that investors investing in portfolios with higher market risks have higher returns due to increase in prices of their shares and received dividend payments (see, for example, Brealey & Mayers, 1996). Higher portfolio market risks come from higher corporate market risks. Here we try to understand what market risks are rewarded by the market at the corporate level presuming that the considered above sustainable corporate development is effective.

It was shown that dividends are paid from free money which a company earns at the market. So, taking extra market risks must result in the growth of free money. Below we derive requirements which must be met to have more money for dividend payments.

Let at a time t_0 a corporate keep a position (ms_0, Q_0) in the area of corporate sustainable development (Fig. 4). This position is characterized by the minimal permissible margin of safety ms_0^* (see (3.4) for $K = Q_0$) and available free money W_0 ($w_0 = \ln W_0$)

$$ms_0^* = (Q_0^2 + 1) / 2Q_0, \qquad (4.1a)$$

$$w_0 /(\alpha_p \sigma_0) = ms_0 - ms_0^* = ms_0 - (Q_0^2 + 1)/(2Q_0). \qquad (4.1b)$$

Suppose that at a time t_1 the corporate position has changed to (ms_1, Q_1) with the minimal permissible margin of safety ms_1^* and the available free money W_1 ($w_1 = \ln W_1$). An initial logarithmic variance of corporate assets consists of the market and individual variances: $\sigma_0^2 = \sigma_m^2 + \sigma_i^2$. Let the individual volatility relate to the market volatility as $\sigma_i = a\sigma_m$, then $\sigma_0^2 = \sigma_m^2 (1 + a^2)$. When the firm changes its position, its market volatility changes to $\sigma_m^1 = b\sigma_m^0,\ b \geq 1$. If the ratio $a = \sigma_i / \sigma_m$ remains the same in the transition than $\sigma_1 = b\sigma_0$, and the margin of safety is

$$ms_1 = (H_0 - DL)/(\alpha_p \sigma_1) = ms_0 / b, \ (4.2a)$$

If that ratio changes to $a_1 = \sigma_i^1 / \sigma_m^1$, than $\sigma_1 = b(1+a_1^2)^{1/2}(1+a_0^2)^{-1/2}\sigma_0 \equiv b'\sigma_0$, and the margin of safety becomes

$$ms_1 = (H_0 - DL)/(\alpha_p \sigma_1) = ms_0 / b' \qquad (4.2b)$$

$$b' = b(1+a_1^2)^{1/2}(1+a_0^2)^{-1/2} \qquad (4.2c)$$

Further we consider the case (4.2a) understanding that we can always extend results on the case (4.2b) using correction (4.2c).

The change in Q is caused by changes in volatility, ROA ($R_1^1 = cR_1^0$), and in parameter C which within one industry characterizes an intensity of competition and capability of the corporate team to resist adverse effects of market fluctuations:

$$Q_1 = \alpha_p C/(R_1^1 \sigma_1) = Q_0/(bc), \qquad (4.3a)$$

$$w_1/(\alpha_p \sigma_0) = b(ms_1 - ms_1^*) = ms_0 - (Q_0^2 + b^2c^2)/(2Q_0 c). \qquad (4.3b)$$

Our goal is to determine conditions for b and c when the corporate position is not worsened by changes of parameters b and c

$$(w_1 - w_0)/(\alpha_p \sigma_0) = (Q_0^2 + 1)/(2Q_0) - (Q_0^2 + b^2c^2)/(2Q_0 c) \geq 0. \quad (4.4)$$

The last inequality can be rewritten as

$$b^2c^2 - (Q_0^2 + 1)c + Q_0^2 \leq 0. \qquad (4.5)$$

To have a meaningful solution to the inequality (4.5), the following conditions must be met

$$1 \leq b \leq \frac{Q_0^2 + 1}{2Q_0}, \qquad (4.6a)$$

$$[ms_0 + (ms_0^2 - 1)^{1/2}]^{-1} \leq Q_0 \leq ms_0 + (ms_0^2 - 1)^{1/2}. \qquad (4.6b)$$

Solving inequality (4.5), we derive a condition for c as a function of b

$$\frac{(Q_0^2 + 1)/b^2}{1 + (1 - A^2)^{1/2}} \leq c \leq \frac{Q_0^2 + 1}{b^2}[1 + (1 - A^2)^{1/2}], \quad (4.7a)$$

$$A \equiv 2Q_0 b/(Q_0^2 + 1). \qquad (4.7c)$$

The graphs for the areas securing improvement in corporate free money are given in Fig. 5 and Fig. 6. From the condition (4.7), it follows immediately that for $Q_0 = 1$ and any $ms_0 > 1$, $b = 1$ and $c = 1$, what means that the position with $Q_0 = 1$ and any $ms_0 > 1$ is the best, and any shift from it reduces available free money.

For other corporate positions, an extra market risk is rewarded only within the limits of one and $Maxb(Q_0, ms_0)$. If for a fixed value of ROA (fixed *c*) the market volatility goes over the maximum, it makes the available corporate free money to diminish. Using the method, one can numerically estimate a threshold for the risk the firm can take in its specific conditions at which decrease in free money will replace its increase.

But how can a firm raise her ROA? There are two ways. The first is to create a highly profitable market niche protected by patents, know-hows, etc. After that the firm for some time can enjoy high monopolistic prices on her production securing high ROA, moderate *C*, and low *Q* value at approximately the same level of the margin of safety *ms*, thus, generating more free money and creating the possibility to pay higher dividends to her shareholders. The second way is to intrude an existing market niche with ROA higher than an industry's mean ROA. The usual reaction from the niche residents to the intervention is to offer resistance to the intruder implying an intensive competition. It is interesting to note that a competition in this model looks similar to the Japanese Sumo wrestling where two wrestlers try to push each other out from the ring. In the market competition the firms try to push each other out from the area of corporate stability preserving high ROA for the survivors. Intensive competition raises market and individual corporate risks. With temporarily low ROA so characteristic for a bitter commercial struggle, they drive fast all the competitors to the boundary of the region of corporate stability until one or several of them leave the region and stop their struggle for a place in the niche and begin a new struggle for their own survival (see the first two sections of the paper).

So, we see that in general the intensity of competition increases as ROA grows from the top to bottom of the area of sustainable development. Most developing firms cannot afford large margins of safety for any ROA and run their business in the narrow boundary layer of the area of corporate stability from time to time leaving it and returning back. When firms leave the area of corporate stability their probability to default increases and they fill themselves distressed. If they stay outside the area for a longer time and at a longer distance, they may come to default. A significant share of distressed firms comes from the top of the area of corporate stability where ROA is low. Of course, all these hypotheses need experimental proofs from market observations.

Now we can interpret the graph in the Fig. 6. For a fixed margin of safety, for any position in the upper part of the area of corporate sustainability there is a position in the bottom part of this area conjugated about a position with $Q = 1$. For example, for a corporate position with $Q = 3$, a conjugated position has $Q = 1/3$ and its ROA is nine times higher. Firm assets in that second position grow very fast, but to secure the same margin of safety the firm must keep big bounded resources hardly leaving anything for free money. The firm belongs to "stars" in the terminology of the Business Consulting Group Growth-Share matrix (Henderson, 1970). When the firm's top management decides that the corporation is big enough, they begin to descend from the area of intensive competition to the areas with lesser ROA and lesser competition (the "star" turns to a "cash cow"). Both for the firms moving to the areas with higher ROA and higher competition, and for the firms moving in the opposite direction, for any change in ROA *c* there is a maximum reasonable risk *Maxb* separating the states securing more free money from those for which an amount of free money begins to decrease. In the upper part of the area of stability close to its boundary, one can find "dogs", and in the middle part of the area one can see the "question marks" in terminology of the BCG matrix.

Fig. 5 and 6 show that increase in the market risk does not always guarantee higher dividend payments as CAPM states, but there is a limit for the market risk getting over which decreases a corporate capability to pay dividends keeping the same margin of safety.

In conclusion we must say that results of this section should be taken with a due caution because they are derived for the case when corporate obligatory payments equal zero: $p(t)=0$. The CAPM is a heuristic model generalizing market observations, and, therefore, taking into account real cases with $p(t)\neq 0$. A solution of the general problem for the corporate financial distress can give real limits of validity of the CAPM.

The model also explains consequences of use of the trial and error method in crisis management when the corporate team tries to subdue the economic distress blindly experimenting with system parameters. To some extent this way of action is specific for small firms trying to compensate a lack of managerial experience with the trial and error method. As a consequence, the asset variance increases faster for small companies compared to medium and large ones managed by professionals. This relation between the corporation size and the growth rate of the asset variance together with a low margin of safety so usual for small companies explains the higher rate of bankruptcies specific for small companies compared to medium and large ones, especially in hard times when adverse effects of market volatility and a negative economic trend strengthen each other ($R_1 = R + m - C/2$, $m < 0$, C is always positive).

Conclusion

The article suggests a quantitative model of a corporate economic distress studying a corporate crisis which lessens corporate ROA at a fixed statistically estimated rate in a special case when all company assets are used to generate a current cash flow (there is no such payments as fixed costs, long-term debts, R&D expenses, etc.). The company's environment is described by normally-distributed, time-invariant, and delta-correlated market fluctuations having a measurable intensity C and trend m. The corporate state is identified with assets $x(t)$, their structure and variance σ^2, company's capability to resist adverse effects of market fluctuations C and periodical order-restoring procedure $F(\sigma,t)$. The crisis is characterized by the time of its onset T_A and rate of ROA decrease w. The recovery program is described by its expected rate of ROA raise u and delay T_1 about the crisis onset. It is shown how the probability of default develops in time at various stages of the distress depending on the environment, company, crisis, and recovery program parameters.

Maybe the most important result of this study is the proof that development of a corporate economic distress and a corporate struggle for survival are dynamic problems essentially depending on proper timing of all measures. Therefore, there is no time-independent criterion for the distance to default or the probability of default. To some extent this result is equivalent to the Black & Cox's solution (1976), however the entire approach used in the paper shows the limitations of the results. Because $p(t)$ is the corporate expenses which do not generate current cash flow, there is no firm for whom $p(t) = 0$ for a sufficiently long time interval. Therefore this solution has mainly a methodological value.

The model joins financial (assets, ROA, taxes, and dividends) and organizational (an organization's structure, personnel motivation, training, management quality, risk avoidance strategy, etc.) characteristics of a company; the last are integrated in the asset variance σ^2, its growth rate *C*, order-restoring procedure $F(\sigma, t)$, and default value $x_{\min}$. Effects of financial or organizational parameters on a corporate business taken separately are widely discussed in management science (see, for example, Brealey & Myers (2000), Robbins (1990)). The model allows quantitative tracing of the contribution to the probability of default of any extra unit of taxes or dividends, any extra points of corporate ROA, and also of changes in organization's parameters.

Statistically monitoring and analyzing the corporate business one can determine

- Time of the crisis onset T_A and rate of ROA decrease *w* caused by the crisis providing for an early crisis detection
- variance of corporate assets σ^2 and its growth rate *C* caused by market fluctuations
- trend *m* in a company industry.

These parameters help determine

- mean and confidence interval for the distance to default if the crisis is left unattended
- marginal delay $T_1^{\max}(u)$ (the point of no return) for starting a recovery program of estimated efficiency *u*. The program launched after this time has a very little chance to succeed
- recovery time T_r and assets $x(T_r)$ and also critical time T_{cr} for which corporate assets have the minimal value x_{cr}
- the probability of corporate default *PRD* as a function of the recovery plan characteristics and problem's parameters
- marginal amount of money that can be withdrawn from the business without exposing it to an extra risk of default.

Measuring corporate, environment, and crisis parameters and estimating characteristics of suggested recovery programs, corporate managers can select for implementation the program providing the highest probability of survival and then estimate its effectiveness in practice. This technique provides managers for the quantitative instrument helping them to improve corporate management in a risky business environment and complex economic conditions. The next step in the development of this model is to increase a model's reality by including nonzero obligatory payments $p(t) > 0$.

References

Altman, E. I. (1968). Financial Ratios, Discriminant Analysis and the Prediction of Corporate Bankruptcy. Journal of Finance, 23, 589 – 609.

Altman, E., & Hotchkiss, E. (2006). Corporate Financial Distress and Bankruptcy: Predict and Avoid Bankruptcy, Analyze and Invest in Distressed Debt (3rd ed.), Hoboken, NJ, USA: John Wiley & Sons.

Asquith, P., Gertner, R., & Scharfstein, D. (1994). Anatomy of Financial Distress: An Explanation of Junk Bond Issuers. Quarterly Journal of Economics, 109, 625 – 658.

Bibeault, D. (1982). Corporate Turnaround: How Managers Turn Losers into Winners. New York, USA: McGraw-Hill.

Black, F., & Cox, J. C. (1976). Valuing Corporate Securities: Some Effects of Bond Indenture Provisions. Journal of Finance, 31, 351 – 367.

Black, F., & Scholes, M. (1972). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), 637 – 654.

Brealey, R. A., & Myers, S. C. (2000). Principles of Corporate Finance. (6th ed.), New York, USA: McGraw-Hill.

R. Geske, R. (1977). The Valuation of Corporate Liabilities as Compound Options. Journal of Financial and Quantitative Analysis, 12, 541 – 552.

Gestel, T. V., Baesens, B., Suykens, J., Poel, D. V., Baestaens, D. E., & Willekens, M. (2006). Bayesian kernel based classification for financial distress detection. European Journal of Operational Research, 172, 979 – 1003.

Gordon, M. J. (1971). Towards a Theory of Financial Distress. Journal of Finance, 26(2), 347 – 356.

Henderson, B. (1970). The Product Portfolio.
https://www.bcgperspectives.com/content/classics/strategy_the_product_portfolio/

Kahya, E., & Theodossiou, P. (1999). Predicting Corporate Financial distress: A Time-Series CUSUM Methodology. Review of Quantitative Finance and Accounting, 13, 323 – 345.

Lando, D. (1998). On Cox Processes and Credit Risky Securities. Review of Derivatives Research, 2, 99 – 120.

Merton, R. C. (1974). On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. Journal of Finance, 2, 449 – 471.

Outecheva, N. (2007). Corporate Financial Distress: An Empirical Analysis of Distress Risk. Ph.D. Dissertation, University of St. Gallen, St. Gallen, Germany.

Pearson, C. M., & Claire, J. A. (1998). Reframing crisis management. Academy of Management Review, 23(1), 59 – 76.

Robbins, S. P. (1990). Organization Theory: The Structure and Design of Organizations. Englewood Cliffs, NJ, USA: Prentice Hall.

Roux-Dufort, C. (2007). A passion for imperfections: revisiting crisis management. In: Pearson, C. M. , C. Roux-Dufort, C., & Claire, J. A. (Eds.), International Handbook of Organizational Crisis Management, (pp. 221 – 252). Thousand Oaks, CA, USA: Sage.

Salchenberger, L. M., Cinar, E. M., & Lash, N. A. (1992). Neural Networks: A New Tool for Predicting Thrift Failures. Decision Sciences, 23, 899 – 916.

Shin, K., & Lee, Y. (2002). A Genetic Algorithm Application in Bankruptcy Prediction Modeling. Expert Systems with Applications, 23(3), 321 – 328.

Gorbenko, A. S., & Strebulaev, I. A. (2010). Temporary versus Permanent Shocks: Explaining Corporate Financial Policies. Review of Financial Studies, 23, 2591 – 2647.

Wruck, K. (1990). Financial Distress, Reorganization, and Organizational Efficiency. Journal of Financial Economics, 27, 419 – 444.

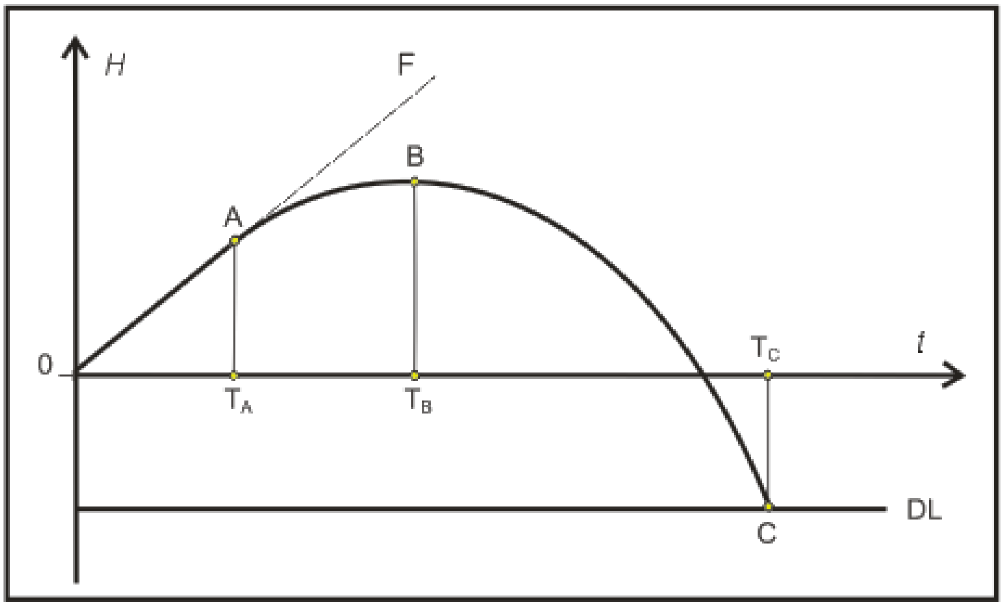

Figure 1: Decline in a mean logarithm of corporate assets *H* over time *t* in a crisis.

Line *0F* is the line of corporate steady progress before the crisis onset at point *A*. T_A is the time of the crisis onset; since that moment corporate ROA linearly declines over time. Point B is the maximum point in the trajectory of the mean logarithm of corporate assets, $ROA(T_B) = 0$. If the crisis is left unattended, a hard default occurs at point C, $x(T_C) = x_{min}$.

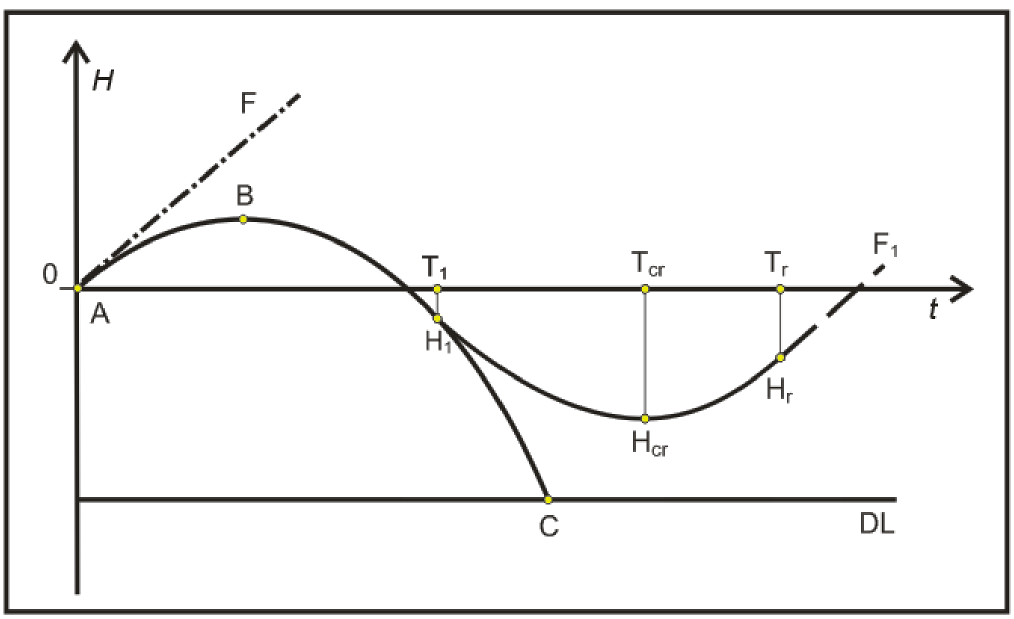

Figure 2: Recovery trajectory for the logarithm of corporate assets passing through the points H_1, H_{cr}, and H_r .

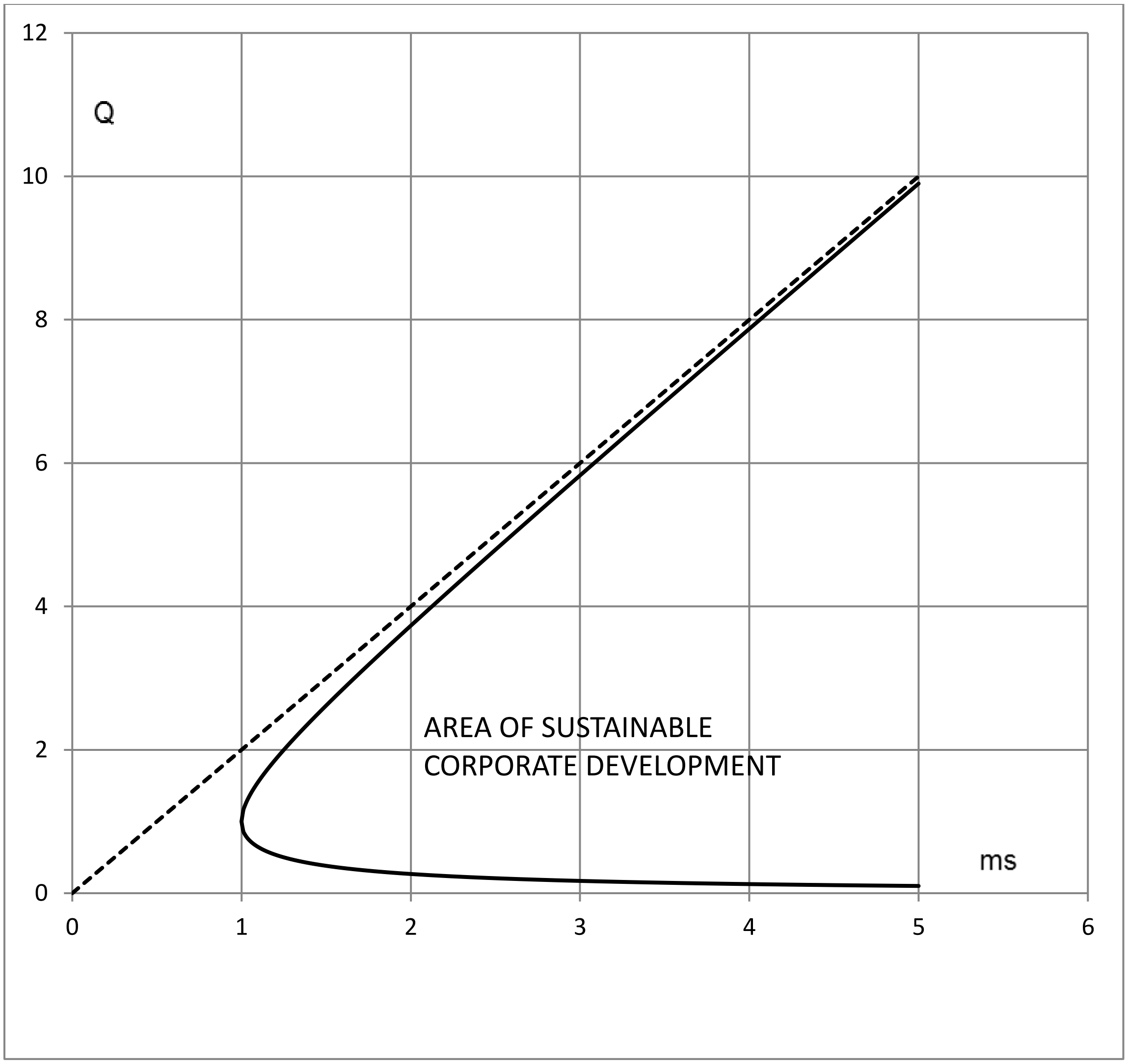

Figure 3: The area of a steady corporate progress, $Q = \alpha_p C / (2R_1\sigma_0)$, $ms = (H_A - DL)/(\alpha_p \sigma_0)$.

The area of a steady corporate progress, where the intensity of the probability of default remains less than 0.05, lies inside the curve.

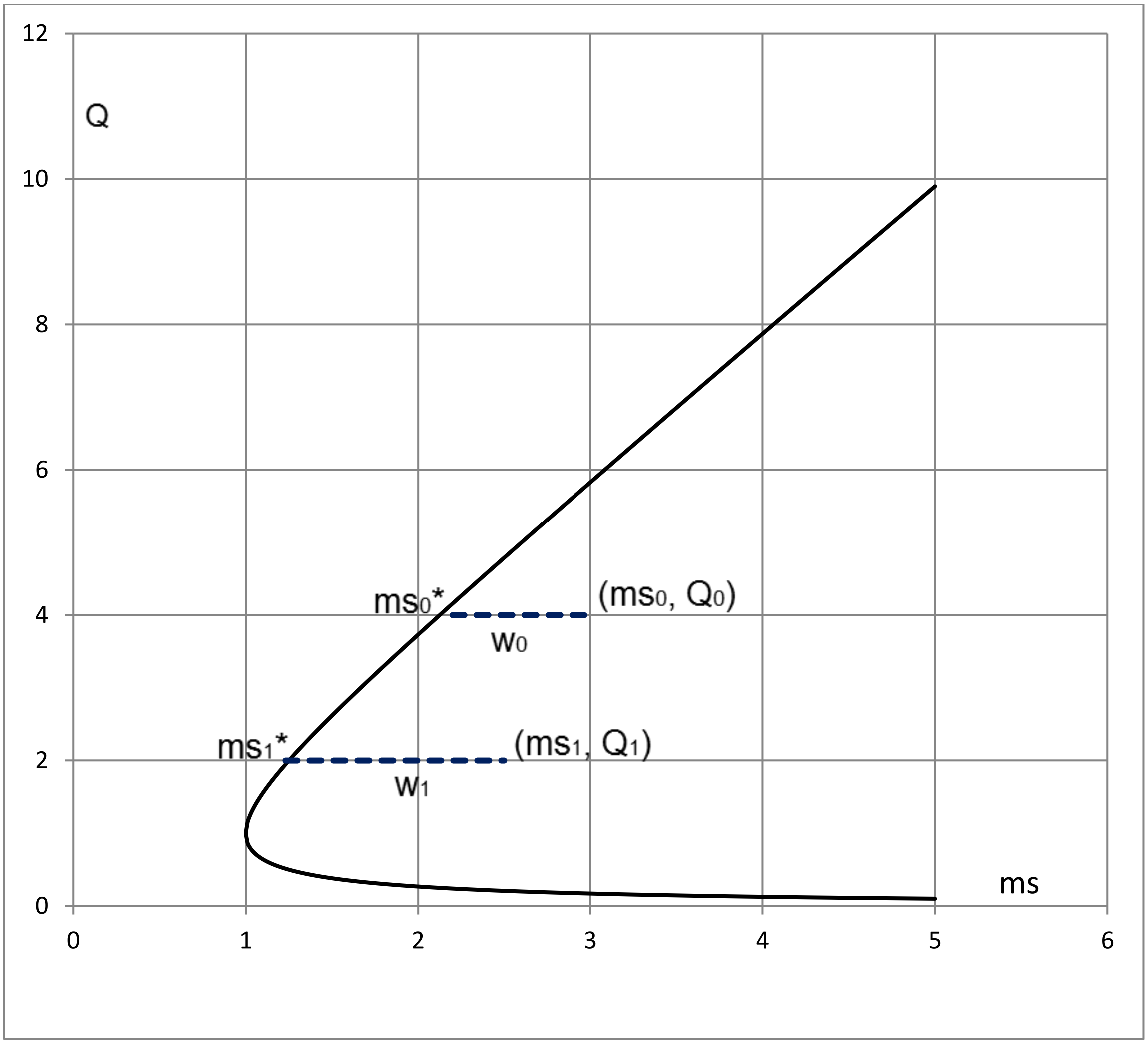

Figure 4: Illustration to calculations of the logarithms of available free money $w_0 = \ln W_0$, $w_1 = \ln W_1$ for different corporate positions (ms_0, Q_0) and (ms_1, Q_1).

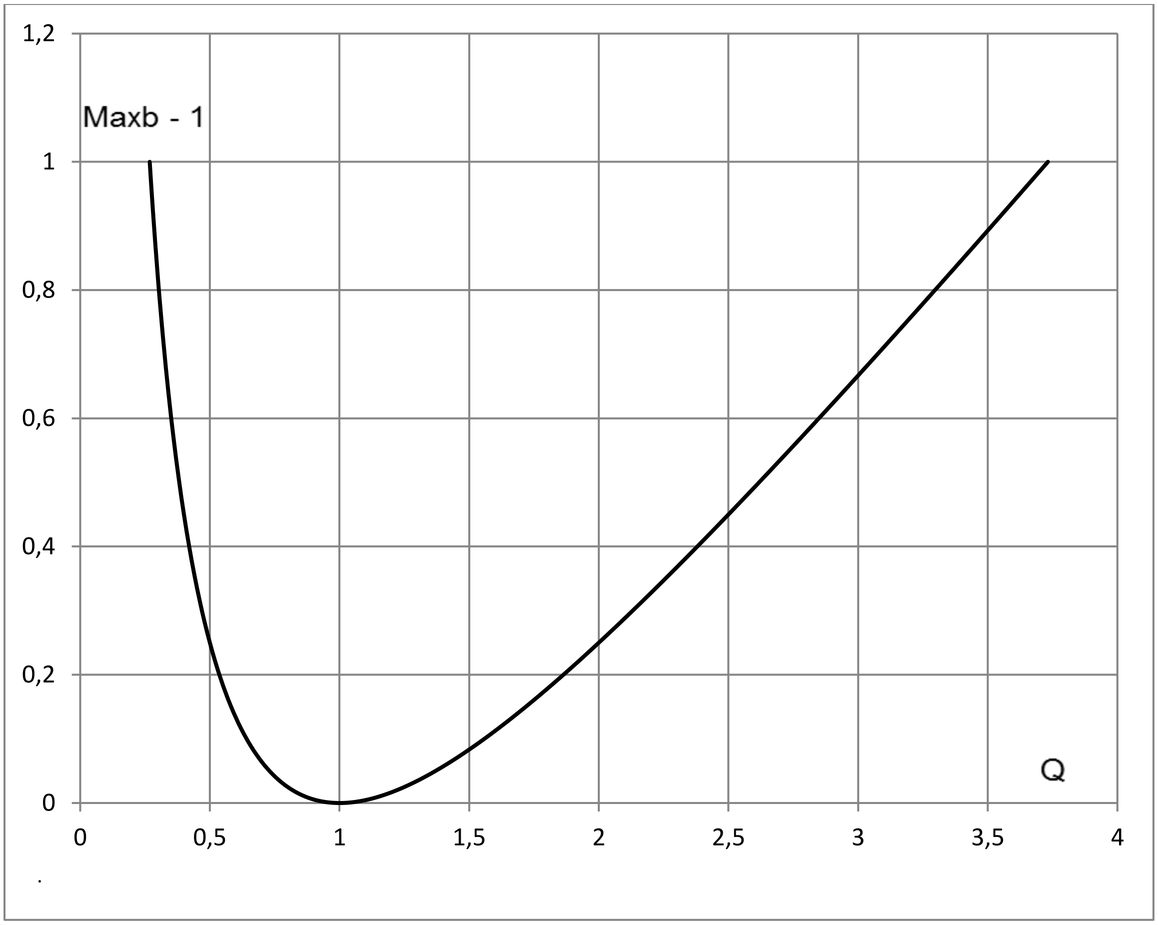

Figure 5: Maximum increase of corporate volatility *Maxb* ($\sigma_m^1 = b\sigma_m^0$) securing increase in free money ($w_1 - w_0 > 0$) keeping the same probability of default as a function of the initial value $Q_0 = \alpha_p C/(2R_1^0\sigma_0^0)$, *ms₀* = 2.

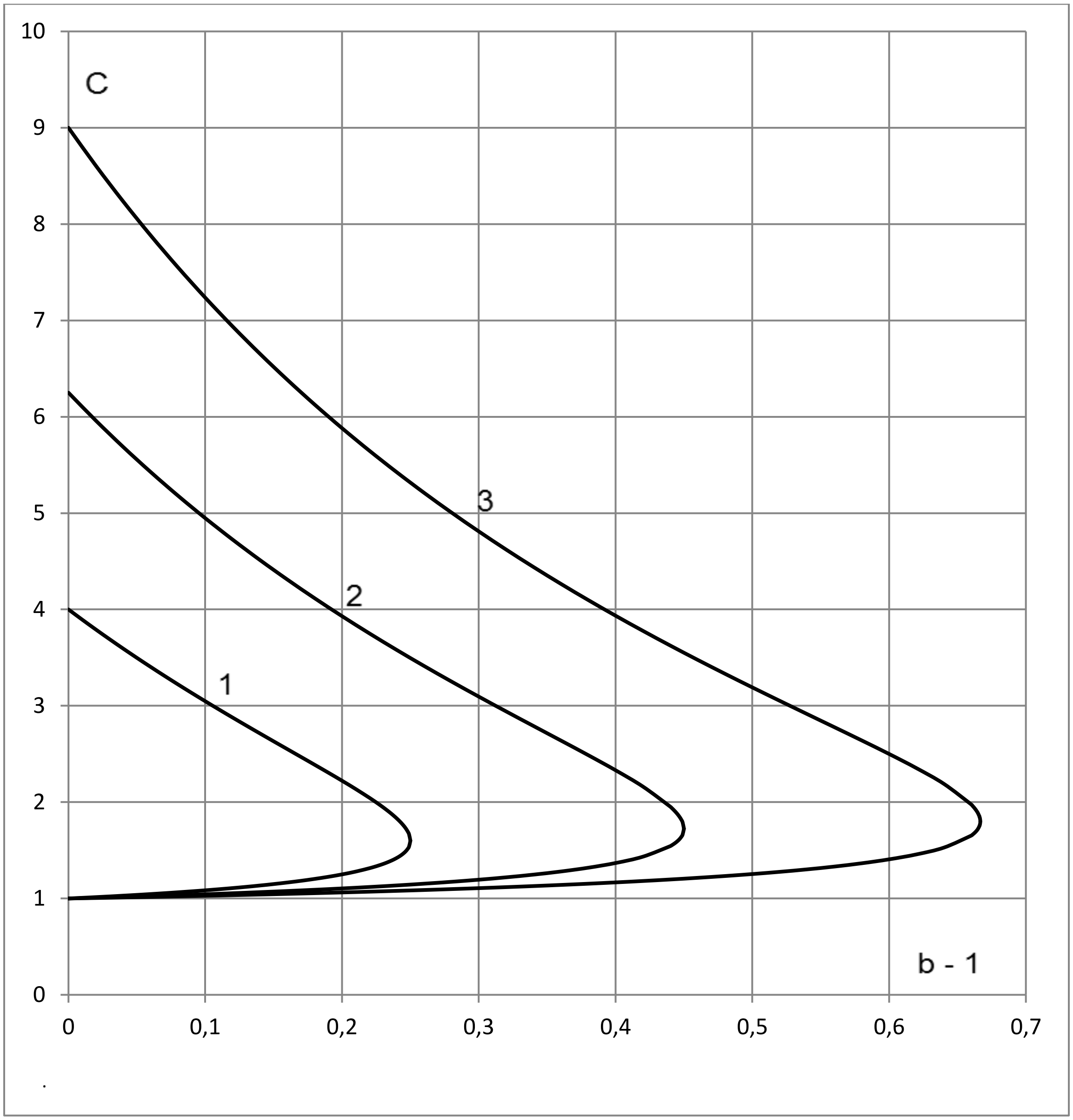

Figure 6: Areas of parameter *c* ($R_1^1 = cR_1^0$) securing increase in free money ($w_1 - w_0 > 0$) and keeping the same probability of default as a function of the volatility growth *b* ($\sigma_m^1 = b\sigma_m^0$) for $ms_0 = 2$ and three values of Q_0: $Q_0 = 2$ (line 1), $Q_0 = 2.5$ (line 2), $Q_0 = 3$ (line 3).